Cubic case

In[15]:=
```
B₀[u_] = (1 - u)^3;
B₁[u_] = 3 (1 - u)^2 u;
B₂[u_] = 3 (1 - u) u^2;
B₃[u_] = u^3;
b₀ = {Cos[φ], -Sin[φ]}; b₁ = {ξ, -η};
b₂ = {ξ, η}; b₃ = {Cos[φ], Sin[φ]};
b30[t_, φ_, ξ_, η_] = Sum[bj Bj[(t + 1) / 2], {j, 0, 3}];
e30[t_, φ_, ξ_, η_] = b30[t, φ, ξ, η][[1]]^2 + b30[t, φ, ξ, η][[2]]^2 - 1 // Simplify;
```

In[20]:=
$$\psi[\text{t\_}] := \frac{1}{16}\left(t^2-1\right)\left(\left(3\,\eta-s\right)^2 t^4+\left(16\,s^2-9\,(\eta+s)^2+9\,(\xi-c)^2\right) t^2+\left(16-\left(3\,\xi+c\right)^2\right)\right);$$

In[21]:=
```
ψ[t] - e30[t, φ, ξ, η] /. {c → Cos[φ], s → Sin[φ]} // Simplify
```

Out[21]= 0

Error function of the best interpolant

In[22]:=
```
Solve[ChebyshevT[6, t] == 0, t]
```

Out[22]=
$$\left\{\left\{t\to-\frac{1}{\sqrt{2}}\right\},\left\{t\to\frac{1}{\sqrt{2}}\right\},\left\{t\to-\frac{1}{2}\sqrt{2-\sqrt{3}}\right\},\left\{t\to\frac{\sqrt{2-\sqrt{3}}}{2}\right\},\left\{t\to-\frac{1}{2}\sqrt{2+\sqrt{3}}\right\},\left\{t\to\frac{\sqrt{2+\sqrt{3}}}{2}\right\}\right\}$$

In[23]:=
$$\chi[\text{t\_}] := 2\left(26-15\sqrt{3}\right)\text{ChebyshevT}\left[6,\frac{\sqrt{2+\sqrt{3}}}{2}\,t\right]\ \text{// Simplify}$$

In[24]:=
$$\left(t^2-1\right)\left(t^2-\left(\sqrt{3}-1\right)^2\right)\left(t^2-\left(2-\sqrt{3}\right)^2\right)-\chi[t]\ \text{// Simplify}$$

Out[24]= 0

Function f

We eliminate η^2

In[25]:=
$$\eta 1=\eta\ \text{/. Solve}\left[\left(1-v^2\right)^2 v^2\,\psi[u]-\left(1-u^2\right)^2 u^2\,\psi[v]==0\ \text{/.}\left\{u\to\sqrt{3}-1,\ v\to 2-\sqrt{3}\right\},\eta\right][[1,1]]\ \text{// FullSimplify}$$

Out[25]=
$$\frac{\left(14+3\sqrt{3}\right)c^2+8\left(-10-6\sqrt{3}+s^2\right)+6\left(2+3\sqrt{3}\right)c\,\xi+27\left(2+\sqrt{3}\right)\xi^2}{24\,s}$$

Equation (2)

In[26]:=
$$\frac{1}{s}\left(\frac{2+\sqrt{3}}{8}\left(3\,\xi+c\right)^2-\xi\,c-3-2\sqrt{3}\right)-\eta 1\ \text{/.}\ s\to\sqrt{1-c^2}\ \text{// Simplify}$$

Out[26]= 0

In[27]:= $$f3 := s^2\,(\psi[u]-\psi[v])\ /.\ \left\{u\to\sqrt{3}-1,\ v\to 2-\sqrt{3},\ \eta\to\frac{1}{s}\left(\frac{2+\sqrt{3}}{8}\,(3\,\xi+c)^2-\xi\,c-3-2\sqrt{3}\right)\right\}\ /.\ s\to\sqrt{1-c^2}\ //\ \text{Simplify}$$

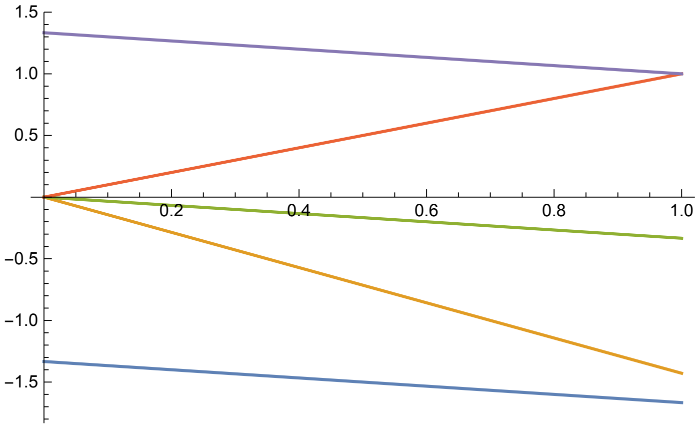
In[28]:= $$f[\xi\_] := 243\,\xi^3-27\,c\left(11-16\sqrt{3}\right)\xi^2-3\left(32\left(1+2\sqrt{3}\right)-3\left(81-32\sqrt{3}\right)c^2\right)\xi-32\left(13+2\sqrt{3}\right)c-\left(163-112\sqrt{3}\right)c^3$$

In[29]:= $$\left(7\sqrt{3}-12\right)\frac{9}{512}\,(\xi-c)\,f[\xi]\ -f3\ //\ \text{Simplify}$$

Out[29]= 0

Position of zeros of function f for c ∈ (0,1)

In[30]:= $$\text{Plot}\left[\left\{-\frac{1}{3}(4+c),\ -\frac{1}{9}\left(8\sqrt{3}-1\right)c,\ -\frac{1}{3}c,\ c,\ \frac{1}{3}(4-c)\right\},\ \{c,0,1\}\right]$$

Out[30]=


In[31]:= $$N\left[7-4\sqrt{3}\right]$$

Out[31]= 0.0717968

In[32]:= $$f\left[-\frac{1}{3}(4+c)\right]-\left(-64\left(7-4\sqrt{3}\right)(1+c)^3\right)\ //\ \text{Simplify}$$

Out[32]= 0

In[33]:= $$f\left[-\frac{1}{9}\left(8\sqrt{3}-1\right)c\right]-\frac{256}{3}\,c\left(1-c^2\right)\ //\ \text{Simplify}$$

Out[33]= 0

In[34]:= $$f\left[-\frac{1}{3}c\right]-\left(-\left(64\,c\left(6+\left(7-4\sqrt{3}\right)c^2\right)\right)\right)\ //\ \text{Simplify}$$

Out[34]= 0

In[35]:= $$f[c]-\left(-256\left(2+\sqrt{3}\right)c\left(1-c^2\right)\right)\ //\ \text{Simplify}$$

Out[35]= 0

In[36]:= $$f\left[\frac{1}{3}(4-c)\right]-64\left(7-4\sqrt{3}\right)(1-c)^3\ //\ \text{Simplify}$$

Out[36]= 0

Zeros of function f for c=0

In[37]:= `Solve[f[ξ] == 0 /. c → 0, ξ]`

Out[37]= $\left\{\{\xi \to 0\}, \left\{\xi \to -\frac{4}{9}\sqrt{2\left(1+2\sqrt{3}\right)}\right\}, \left\{\xi \to \frac{4}{9}\sqrt{2\left(1+2\sqrt{3}\right)}\right\}\right\}$

In[38]:= $\left\{-\frac{1}{3}(4+c), -\frac{4}{9}\sqrt{2\left(1+2\sqrt{3}\right)}, -\frac{1}{9}\left(8\sqrt{3}-1\right)c\right\}$ `/. c → 0 // N`

Out[38]= {-1.33333, -1.328, 0.}

In[39]:= $\left\{c, \frac{4}{9}\sqrt{2\left(1+2\sqrt{3}\right)}, \frac{1}{3}(4-c)\right\}$ `/. c → 0 // N`

Out[39]= {0., 1.328, 1.33333}

Necessary condition on p to be good approximant of c (d) is x(0) > 0 (x(0) < 0)

In[40]:= `b30[0, φ, ξ, η][[1]]` $-\frac{1}{4}(3\xi+c)$ `/. c → Cos[φ] // Simplify`

Out[40]= 0

In[41]:= `Coefficient[ψ[t], t, 0]` $-\left(-\frac{1}{16}\left(16-(3\xi+c)^2\right)\right)$ `// Simplify`

Out[41]= 0

Parameters for the best interpolant when c=1, i.e., 2*π*-*φ*=2*π*

In[42]:= `Solve[`$(1-v^2)^2 v^2 \psi[u] - (1-u^2)^2 u^2 \psi[v] == 0$ `/.` $\{u \to \sqrt{3}-1, v \to 2-\sqrt{3}, c \to 1, s \to 0\}$`, ξ] // Simplify`

Out[42]= $\left\{\{\xi \to 1\}, \left\{\xi \to \frac{1}{9}\left(1-8\sqrt{3}\right)\right\}\right\}$

In[43]:= `Solve[ψ[u] == 0 /.` $\left\{u \to \sqrt{3}-1, v \to 2-\sqrt{3}, c \to 1, s \to 0, \xi \to -\frac{1}{9}\left(8\sqrt{3}-1\right)\right\}$`, η] // Simplify`

Out[43]= $\left\{\left\{\eta \to -\frac{4}{9}\sqrt{38+22\sqrt{3}}\right\}, \left\{\eta \to \frac{4}{9}\sqrt{38+22\sqrt{3}}\right\}\right\}$

Table of best interpolants of c with corresponding errors

In[44]:= `angles = ` $\left\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{8}, \frac{\pi}{12}\right\}$;

In[45]:=
```
sez3 = {{"φ", "ξ", "η", "error"}};
For[i = 1, i ≤ Length[angles], i++, φ0 = angles[[i]];
 ξ0 = Select[ξ /. NSolve[f[ξ] == 0 /. c → Cos[φ0], ξ, Reals, WorkingPrecision → 30],
    # > Cos[φ0] && # < 1/3 (4 + Cos[φ0]) &][[1]];
 η0 = 1/Sin[φ0] ((2 + √3)/8 (3 ξ0 + Cos[φ0])^2 - ξ0 Cos[φ0] - 3 - 2 √3);
 AppendTo[sez3, {φ, NumberForm[ξ0, {6, 5}], NumberForm[η0, {5, 5}],
    ScientificForm[N[-e30[0, φ, ξ0, η0]], 6]} /. φ → angles[[i]]]];
```

In[46]:=
```
Grid[sez3, Frame → All]
```

Out[46]=

φ	ξ	η	error
$\frac{\pi}{2}$	1.32800	0.94046	7.97742×10^{-3}
$\frac{\pi}{3}$	1.16617	0.47494	7.50902×10^{-4}
$\frac{\pi}{4}$	1.09754	0.31523	1.36878×10^{-4}
$\frac{\pi}{6}$	1.04465	0.19043	1.22221×10^{-5}
$\frac{\pi}{8}$	1.02537	0.13762	2.18815×10^{-6}
$\frac{\pi}{12}$	1.01136	0.08926	1.92912×10^{-7}

You can draw the best interpolant of c, the corresponding error function and the curvature for arbitrary angle $\varphi 0 \in (0, \frac{\pi}{2}]$

In[47]:=
```
φ0 = π/3;
ξ0 = Select[ξ /. NSolve[f[ξ] == 0 /. c → Cos[φ0], ξ, Reals, WorkingPrecision → 30],
    # > Cos[φ0] && # < 1/3 (4 + Cos[φ0]) &][[1]];
η0 = 1/Sin[φ0] ((2 + √3)/8 (3 ξ0 + Cos[φ0])^2 - ξ0 Cos[φ0] - 3 - 2 √3);
GraphicsRow[
  {Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0}, PlotStyle → {Blue, Dashed},
     Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}], ParametricPlot[
     {b30[t, φ0, ξ0, η0][[1]], b30[t, φ0, ξ0, η0][[2]]}, {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2, b3}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[e30[t, φ0, ξ0, η0], {t, -1, 1}],
   Plot[Evaluate[(D[b30[t, φ0, ξ0, η0][[1]], t] × D[b30[t, φ0, ξ0, η0][[2]], {t, 2}] -
        D[b30[t, φ0, ξ0, η0][[2]], t] × D[b30[t, φ0, ξ0, η0][[1]], {t, 2}]) /
      (D[b30[t, φ0, ξ0, η0][[1]], t]^2 + D[b30[t, φ0, ξ0, η0][[2]], t]^2)^(3/2)],
    {t, -1, 1}]}] /. {φ → φ0, ξ → ξ0, η → η0}
```

Out[50]=

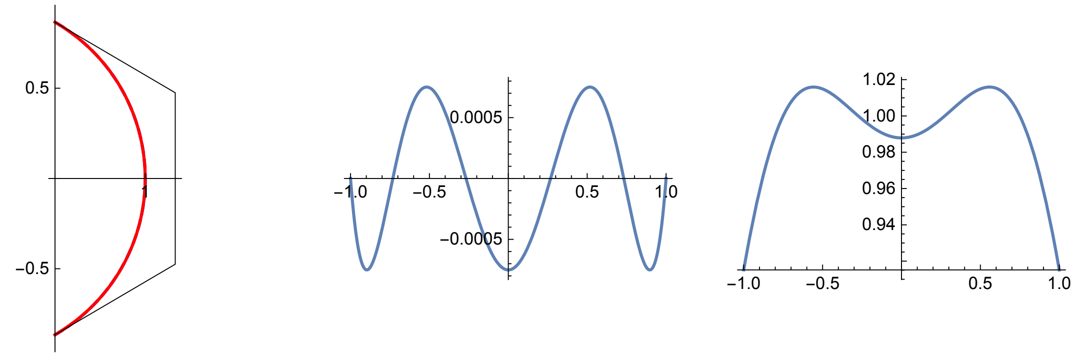


Table of best interpolants of d with corresponding errors

In[51]:=
```
angles = {π/2, π/3, π/4, π/6, π/8, π/12};
```

In[52]:=
```
sez3 = {};
For[i = 1, i ≤ Length[angles], i++, φ0 = angles[[i]];
 ξ0 = Select[ξ /. NSolve[f[ξ] == 0 /. c → Cos[φ0], ξ, Reals, WorkingPrecision → 30],
    # > -1/3 (4 + Cos[φ0]) && # < -1/9 (8 √3 - 1) Cos[φ0] &][[1]];
 η0 = 1/Sin[φ0] ((2 + √3)/8 (3 ξ0 + Cos[φ0])^2 - ξ0 Cos[φ0] - 3 - 2 √3);
 PrependTo[sez3, {π - φ, NumberForm[ξ0, {6, 5}], NumberForm[η0, {5, 5}],
    ScientificForm[N[-e30[0, φ, ξ0, η0]], 6]} /. φ → angles[[i]]]];
PrependTo[sez3, {"φ", "ξ", "η", "error"}];
```

In[54]:=
```
Grid[sez3, Frame → All]
```

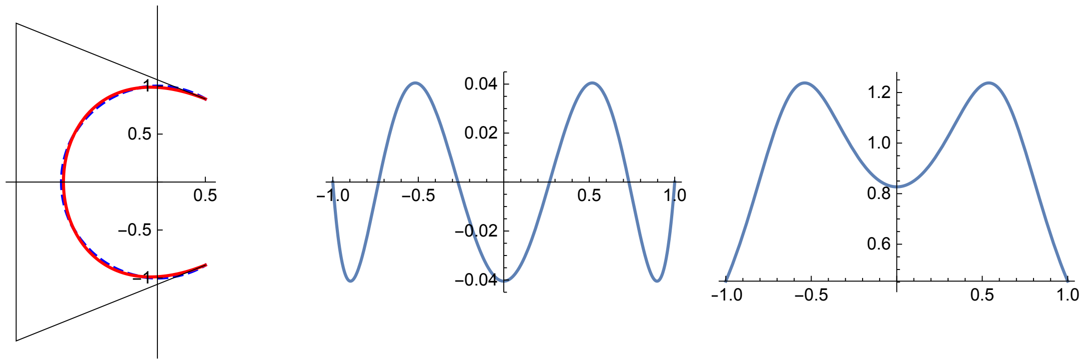
Out[54]=

φ	ξ	η	error
$\frac{11\pi}{12}$	-1.50262	3.24480	2.15914×10^{-1}
$\frac{7\pi}{8}$	-1.52161	2.94230	1.71465×10^{-1}
$\frac{5\pi}{6}$	-1.52964	2.65270	1.33756×10^{-1}
$\frac{3\pi}{4}$	-1.51679	2.11990	7.68372×10^{-2}
$\frac{2\pi}{3}$	-1.47274	1.65620	4.04723×10^{-2}
$\frac{\pi}{2}$	-1.32800	0.94046	7.97742×10^{-3}

You can draw the best interpolant of d, the corresponding error function and the curvature for arbitrary angle $\varphi 0 \in [0, \frac{\pi}{2}]$

In[55]:=
```
φ0 = π / 3;
ξ0 = Select[ξ /. NSolve[f[ξ] == 0 /. c → Cos[φ0], ξ, Reals, WorkingPrecision → 30],
     # > -1/3 (4 + Cos[φ0]) && # < -1/9 (8 √3 - 1) Cos[φ0] &][[1]];
η0 = 1/Sin[φ0] ((2 + √3)/8 (3 ξ0 + Cos[φ0])^2 - ξ0 Cos[φ0] - 3 - 2 √3);
GraphicsRow[
  {Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0 - 2 π}, PlotStyle → {Blue, Dashed},
     Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}, AxesOrigin → {0, 0}],
    ParametricPlot[{b30[t, φ0, ξ0, η0][[1]], b30[t, φ0, ξ0, η0][[2]]}, {t, -1, 1},
     PlotStyle → Red], ListPlot[{b0, b1, b2, b3}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[e30[t, φ0, ξ0, η0], {t, -1, 1}],
   Plot[Evaluate[- (D[b30[t, φ0, ξ0, η0][[1]], t] × D[b30[t, φ0, ξ0, η0][[2]], {t, 2}] -
          D[b30[t, φ0, ξ0, η0][[2]], t] × D[b30[t, φ0, ξ0, η0][[1]], {t, 2}]) /
       (D[b30[t, φ0, ξ0, η0][[1]], t]^2 + D[b30[t, φ0, ξ0, η0][[2]], t]^2)^(3/2)],
    {t, -1, 1}]}] /. {φ → φ0, ξ → ξ0, η → η0}
```

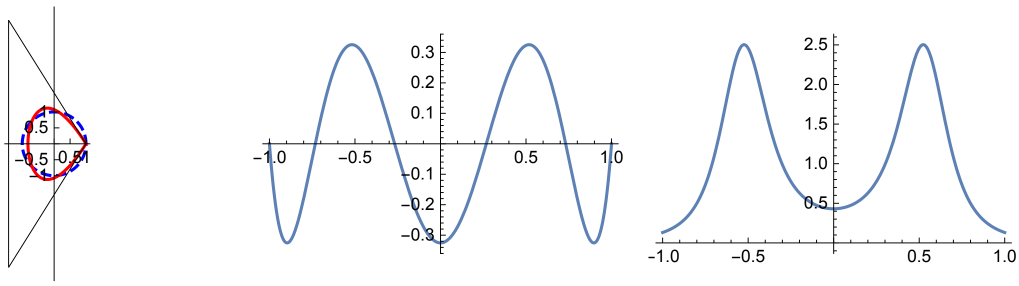
Out[58]=



The best interpolant, the corresponding error function and the curvature for whole unite circle

In[59]:=
```
φ0 = 0;
ξ0 = -1/9 (8 √3 - 1);
η0 = 4/9 √(38 + 22 √3);
GraphicsRow[
  {Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0 - 2 π}, PlotStyle → {Blue, Dashed},
     Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}, AxesOrigin → {0, 0}],
    ParametricPlot[{b30[t, φ0, ξ0, η0][[1]], b30[t, φ0, ξ0, η0][[2]]}, {t, -1, 1},
     PlotStyle → Red], ListPlot[{b0, b1, b2, b3}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[e30[t, φ0, ξ0, η0], {t, -1, 1}],
   Plot[Evaluate[- (D[b30[t, φ0, ξ0, η0][[1]], t] × D[b30[t, φ0, ξ0, η0][[2]], {t, 2}] -
          D[b30[t, φ0, ξ0, η0][[2]], t] × D[b30[t, φ0, ξ0, η0][[1]], {t, 2}]) /
       (D[b30[t, φ0, ξ0, η0][[1]], t]^2 + D[b30[t, φ0, ξ0, η0][[2]], t]^2)^(3/2)],
    {t, -1, 1}]}] /. {φ → φ0, ξ → ξ0, η → η0}
```

Out[62]=



Quartic case

In[63]:=
```
B₀[u_] = (1 - u)^4;
B₁[u_] = 4 (1 - u)^3 u;
B₂[u_] = 6 (1 - u)^2 u^2;
B₃[u_] = 4 (1 - u) u^3;
B₄[u_] = u^4;
b₀ = {Cos[φ], -Sin[φ]}; b₁ = {α, -β};
b₂ = {γ, 0}; b₄ = {Cos[φ], Sin[φ]}; b₃ = {α, β};
b40[t_, φ_, α_, β_, γ_] = Sum[bⱼ Bⱼ[(t + 1) / 2], {j, 0, 4}]; // Simplify;
e40[t_, φ_, α_, β_, γ_] =
  b40[t, φ, α, β, γ][[1]]^2 + b40[t, φ, α, β, γ][[2]]^2 - 1 // Simplify;
```

In[68]:= $\psi[t\_] := -1 + \frac{1}{64}\left(4\left(1-t^4\right)\alpha + 3\left(1-t^2\right)^2\gamma + \left(1+6t^2+t^4\right)c\right)^2 + \frac{1}{4}t^2\left(2\left(1-t^2\right)\beta + \left(1+t^2\right)s\right)^2;$

In[69]:=
```
ψ[t] - e40[t, φ, α, β, γ] /. {c → Cos[φ], s → Sin[φ]} // Simplify
```

Out[69]= 0

Error function of the best interpolant

In[70]:=
```
Solve[ChebyshevT[8, t] == 0, t]
```

Out[70]= $\left\{\left\{t \to -\frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{2}}}\right\}, \left\{t \to \frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{2}}}\right\},\right.$
$\left\{t \to -\frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}}\right\}, \left\{t \to \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}}\right\}, \left\{t \to -\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}\right\},$
$\left.\left\{t \to \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}\right\}, \left\{t \to -\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}\right\}, \left\{t \to \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}\right\}\right\}$

In[71]:= $\chi[t\_] := \frac{2}{\left(2+\sqrt{2+\sqrt{2}}\right)^4}\,\mathrm{ChebyshevT}\left[8, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}\; t\right];$

In[72]:= $u_1 = \sqrt{2\left(2+\sqrt{2}\right)} - 1 - \sqrt{2};\; u_2 = \sqrt{2+\sqrt{2}} - 1;\; u_3 = 1 + \sqrt{2} - \sqrt{2+\sqrt{2}};$

In[73]:= $\chi[t] - \left(t^2-1\right)\left(t^2-u_1^2\right)\left(t^2-u_2^2\right)\left(t^2-u_3^2\right)$ // Simplify

Out[73]= 0

In[74]:=
```
σ1 = (1 - u1^2) + (1 - u2^2) + (1 - u3^2);
σ2 = (1 - u1^2) (1 - u2^2) + (1 - u1^2) (1 - u3^2) + (1 - u2^2) (1 - u3^2);
σ3 = (1 - u1^2) (1 - u2^2) (1 - u3^2);
```

In[77]:=
```
Grid[{{"σ1", "σ2", "σ3"}, {FullSimplify[σ1], ToRadicals[FullSimplify[σ2]],
   ToRadicals[FullSimplify[σ3]]}, {N[σ1], N[σ2], N[σ3]}}, Frame → All]
```

Out[77]=

σ_1	σ_2	σ_3
$4\left(-3-2\sqrt{2}+\left(2+\sqrt{2}\right)^{3/2}\right)$	$2\left(89+63\sqrt{2}-2\sqrt{2\left(1970+1393\sqrt{2}\right)}\right)$	$4\left(-185-131\sqrt{2}+\sqrt{2\left(34\,282+24\,241\sqrt{2}\right)}\right)$
1.92087	1.11348	0.183483

Function f

We eliminate the parameter β and we get the equation (4)

In[78]:=

$$\text{term4} = 64\left(\frac{u_2^2 u_3^2 \psi[u_1]}{(u_2^2-u_1^2)(u_1^2-u_3^2)(1-u_1^2)} + \frac{u_1^2 u_3^2 \psi[u_2]}{(u_3^2-u_2^2)(u_2^2-u_1^2)(1-u_2^2)} + \frac{u_1^2 u_2^2 \psi[u_3]}{(u_1^2-u_3^2)(u_3^2-u_2^2)(1-u_3^2)}\right);$$

$$\text{term4simpl} = 64 + u_1^2 u_2^2 u_3^2 (4\alpha - 3\gamma - c)^2 - (4\alpha + 3\gamma + c)^2;$$

In[79]:= term4 - term4simpl /. $s \to \sqrt{1-c^2}$ // Simplify

Out[79]= 0

In[80]:= Solve[4 α - 3 γ - c == x && 4 α + 3 γ + c == y, {α, γ}]

Out[80]= $\left\{\left\{\alpha \to \frac{x+y}{8}, \gamma \to \frac{1}{6}(-2c - x + y)\right\}\right\}$

Equation (5)

In[81]:=

$$\text{term5} = \left(\left(u_3^2 (1-u_3^2)^2 - u_2^2 (1-u_2^2)^2\right)\psi[u_1] + \left(u_1^2 (1-u_1^2)^2 - u_3^2 (1-u_3^2)^2\right)\psi[u_2] + \left(u_2^2 (1-u_2^2)^2 - u_1^2 (1-u_1^2)^2\right)\psi[u_3]\right) - \frac{\sigma_2}{64}(u_1^2-u_2^2)(u_2^2-u_3^2)(u_3^2-u_1^2)(64 + u_1^2 u_2^2 u_3^2 x^2 - y^2) \;/.\; \left\{\alpha \to \frac{x+y}{8}, \gamma \to \frac{1}{6}(-2c - x + y)\right\};$$

$$\text{term5simpl} = -1 + \frac{1}{128}\sigma_3 x^2 + \frac{1}{8}(x+y)\,c + \beta\, s;$$

In[83]:= term5 - 2 $(u_1^2-u_2^2)(u_2^2-u_3^2)(u_3^2-u_1^2)(\sigma_2 - \sigma_1 + 1)$ term5simpl /. $s \to \sqrt{1-c^2}$ // Simplify

Out[83]= 0

Equation before (6)

In[84]:=
$$\text{term6a} = \left(\frac{\psi[u_1]}{1-u_1^2} + \frac{\psi[u_2]}{1-u_2^2} + \frac{\psi[u_3]}{1-u_3^2}\right) + \frac{1}{64}\sigma_1\left(64 + u_1^2 u_2^2 u_3^2 x^2 - y^2\right) - \left(\sigma_1^2 - 3\sigma_1 - 2\sigma_2 + 6\right)\left(-1 + \frac{1}{128}\sigma_3 x^2 + \frac{1}{8}c\,(x+y) + \beta s\right) \,/.\, \left\{\alpha \to \frac{x+y}{8},\ \gamma \to \frac{1}{6}\left(-2c - x + y\right)\right\};$$
$$\text{term6asimpl} = 96 - 4xy - 128\beta^2 + 16c\,(x-y) + 32c^2 - \left(4 - 2\sigma_1 + \sigma_3\right)x^2;$$

In[86]:=
$$\text{term6a} - \frac{1}{128}\left(\sigma_1^2 - 2\sigma_2 - \sigma_1\right)\text{term6asimpl} \,/.\, s \to \sqrt{1-c^2} \ //\ \text{Simplify}$$

Out[86]= 0

Equation (6)

In[87]:=
$$\text{term6} = \frac{1}{128}s^2\,\text{term6asimpl} - \left(\left(1 - \frac{1}{128}\sigma_3 x^2 - \frac{1}{8}c\,(x+y)\right)^2 - \beta^2 s^2\right) - \frac{c^2}{64}\text{term4simpl} \,/.\, \left\{\alpha \to \frac{x+y}{8},\ \gamma \to \frac{1}{6}\left(-2c - x + y\right)\right\};$$
$$\text{term6simpl} = \left(-\frac{x}{32} + \frac{c}{8} + \frac{c^3}{8} - \frac{1}{512}\sigma_3 c\,x^2\right)y - \frac{1}{4}\left(\left(\frac{1}{64}\sigma_3 x^2 + \frac{1}{4}c\,x - s^2\right)^2 - \frac{1}{16}\left(1 - \sigma_2 + 2\sigma_3\right)c^2 x^2 + \frac{1}{16}\left(2 - \sigma_1\right)x^2 - c\left(x - 8c\right)\right);$$

In[89]:=
$$\text{term6} - \text{term6simpl} \,/.\, s \to \sqrt{1-c^2} \ //\ \text{Simplify}$$

Out[89]= 0

In[90]:=
$$f[x\_] := \frac{1}{16}\left(\left(\frac{1}{64}\sigma_3 x^2 + \frac{1}{4}c\,x - 1 + c^2\right)^2 - \frac{1}{16}\left(1 - \sigma_2 + 2\sigma_3\right)c^2 x^2 + \frac{1}{16}\left(2 - \sigma_1\right)x^2 - c\left(x - 8c\right)\right)^2 - \left(64 + u_1^2 u_2^2 u_3^2 x^2\right)\left(\frac{x}{32} - \frac{c}{8} - \frac{c^3}{8} + \frac{1}{512}c\,\sigma_3 x^2\right)^2$$

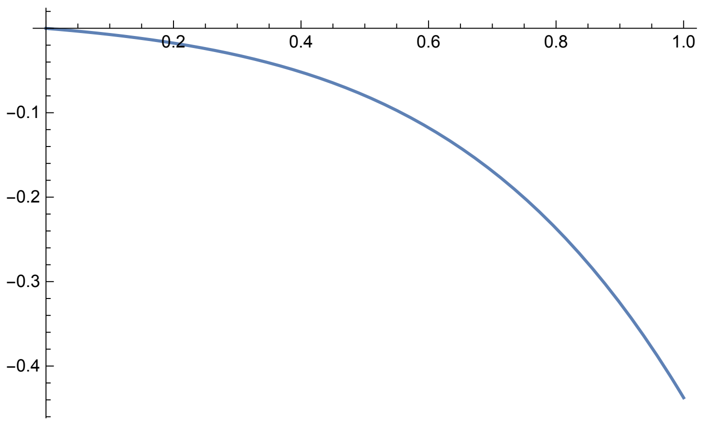
Lemma 3

$f\left[-(1-c)^2\right] < 0$

In[91]:= PolynomialRemainder$\left[f\left[-(1-c)^2\right],\ (1-c)^4,\ c\right]$

Out[91]= 0

In[92]:= Plot$\left[\text{Evaluate}\left[\text{PolynomialQuotient}\left[f\left[-(1-c)^2\right],\ (1-c)^4,\ c\right]\right],\ \{c, 0, 1\}\right]$

Out[92]=


In[93]:= `PolynomialQuotient[f[-(1-c)^2], (1-c)^4, c] /. c → 0 // Simplify // N`

Out[93]= -0.000106356

$f[0] > 0$

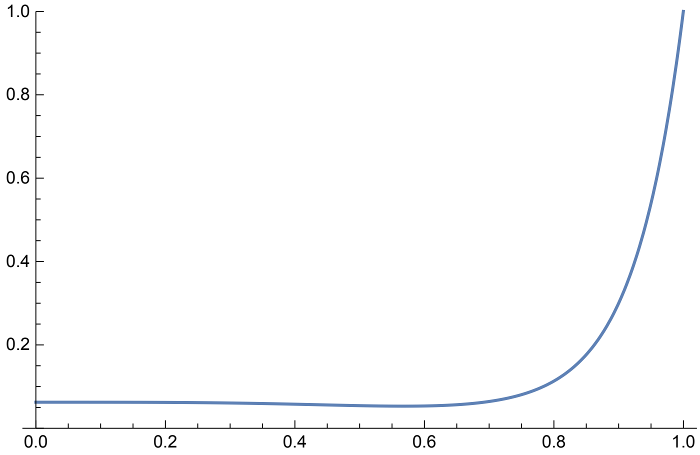
In[94]:= `f[0] // Simplify`

Out[94]= $\frac{1}{16}\left(-1+c^2\right)^4$

$f\left[4\,c\,\left(1+c^2\right)\right] > 0$

In[95]:= `Plot[f[4 c (1 + c^2)], {c, 0, 1}, PlotRange → {0, 1}]`

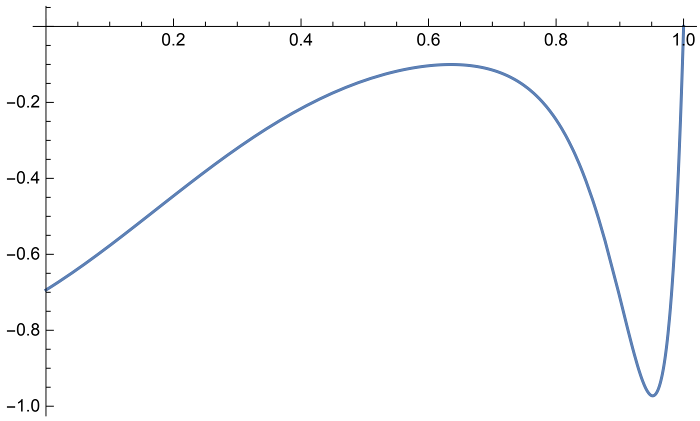
Out[95]=



$f\left[\left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right)\left(1+c^4\right)^2\right] < 0$

In[96]:= `Plot[f[(6 - 3 Sqrt[2] + Sqrt[68 - 46 Sqrt[2]]) (1 + c^4)^2], {c, 0, 1}]`

Out[96]=



In[97]:= `f[x] /. c → 1 /. x → 4 (6 - 3 Sqrt[2] + Sqrt[68 - 46 Sqrt[2]]) // FullSimplify`

Out[97]= 0

Function f is increasing on $(-(1-c)^2, 0)$

In[98]:=
```
g[t_] := f[-t (1 - c)^2];
g1[t_] := 1/16 (1 + c)^2 (5 + c^2) c +
   1/128 (12 - 56 c^2 + 28 c^4 + 2 σ1 (1 - c^2) (1 + 5 c^2 + 2 c^4) + 2 σ2 (1 - c^2)^2 c^2 + σ3 (1 - c^2)^3) t +
   3/1024 (4 σ1 (1 - 3 c^2) + 4 σ2 (2 - c^2 + c^4) + (1 - c^2) (-16 + (-9 + 5 c^2) σ3)) (1 - c)^2 c t^2;
g2[t_] := (5/2 σ3 - σ2 c)^2 t^3;
g3[t_] :=
  (σ2 (32 (1 - c^2) + σ3 (5 - 8 c + 8 c^3) c) - 1/4 σ3 (64 (1 - 5 c^2 + 2 c^4) + σ3 (37 - 24 c^2 + 12 c^4)) +
      8 σ1 (2 σ2 c^2 - σ3 (1 + c^2 - 2 c^4)) - 8 σ1^2 - (c^2 + 8 c^4) σ2^2) t^3 -
   5/8 σ3 (8 σ2 + 5 c^2 σ3) (1 - c)^2 c t^4 - 3/64 σ3^2 (4 + 4 c^2 σ1 + c^2 σ3) (1 - c)^4 t^5 - 1/4096 σ3^4 (1 - c)^8 t^7;
g4[t_] := 5/8 σ3 (4 σ1 + 4 c^2 σ2 + 3 σ3) (1 - c)^2 c t^4 +
   3/64 σ3^2 (2 σ1 + 2 c^2 σ2 + σ3) (1 - c)^4 t^5 + 7/512 σ3^3 (1 - c)^6 c t^6;
```

In[102]:=
```
-((1 - c)^4 g1[t] + 1/8192 (1 - c)^8 (g2[t] + g3[t] + g4[t])) - D[g[t], t] /.
   u1^2 u2^2 u3^2 → (1 - σ1 + σ2 - σ3) /. s → Sqrt[1 - c^2] // Simplify
```

Out[102]= 0

$g_2[t] > 0$

In[103]:= `g2[t] // N`

Out[103]= $(0.458708 - 1.11348\, c)^2\, t^3$

$g_4[t] \geq 0$

In[104]:= `g4[t] // N`

Out[104]= $0.114677\, (1. - 1.\, c)^2\, c\, (8.23392 + 4.4539\, c^2)\, t^4 + 0.0015781\, (1. - 1.\, c)^4\, (4.02522 + 2.22695\, c^2)\, t^5 + 0.0000844536\, (1. - 1.\, c)^6\, c\, t^6$

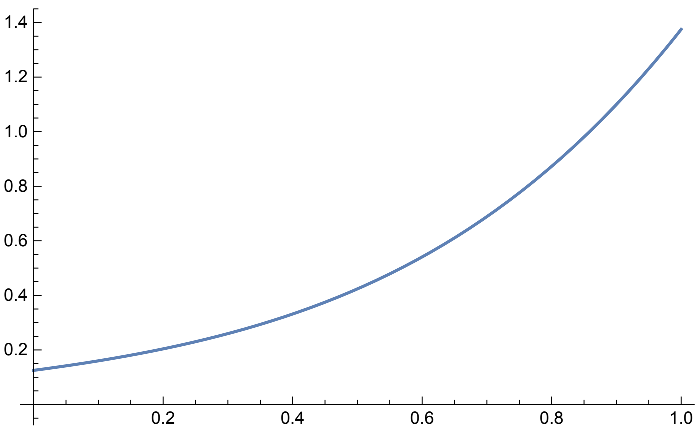
$g_1[t] \geq 0$

In[105]:= `CoefficientList[g1[t], t][[1]]`

Out[105]= $\frac{1}{16} c\, (1 + c)^2\, (5 + c^2)$

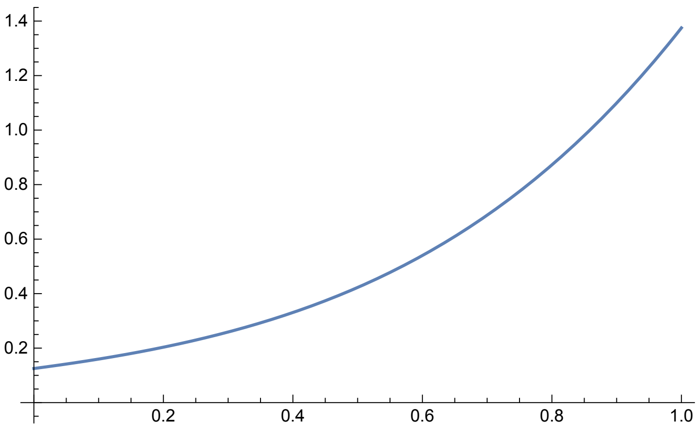
In[106]:= `Plot[Total[CoefficientList[g1[t], t][[1 ;; 2]]] // Simplify, {c, 0, 1}]`

Out[106]=



In[107]:=
```
Plot[Total[CoefficientList[g₁[t], t][[1 ;; 3]]] // Simplify, {c, 0, 1}]
```

Out[107]=



$g_3[t] \geq 0$

In[108]:=
```
Coefficient[g₃[t], t^4] // N
Coefficient[g₃[t], t^5] // N
Coefficient[g₃[t], t^7] // N
```

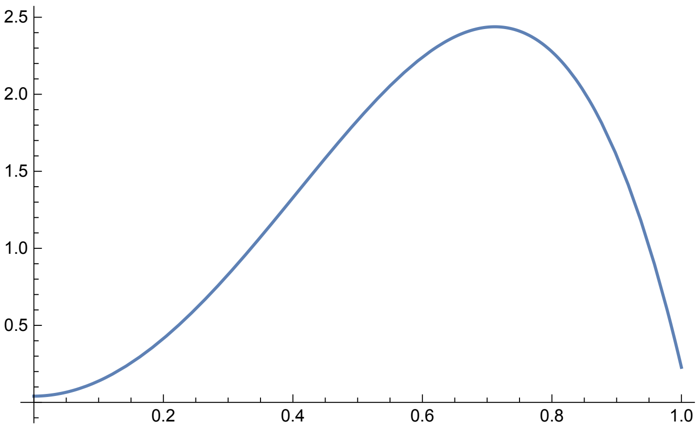
Out[108]= $-0.114677\,(1.-1.\,c)^2\,c\,(8.9078+0.917417\,c^2)$

Out[109]= $-0.0015781\,(1.-1.\,c)^4\,(4.+7.86695\,c^2)$

Out[110]= $-2.76711\times10^{-7}\,(1.-1.\,c)^8$

In[111]:=
```
Plot[g₃[1], {c, 0, 1}]
```

Out[111]=



Function f is increasing on (0,4 c^2)

In[112]:=
$$g_1[x\_] = -\frac{1}{2^{24}}\left(21\,\sigma_3^4\,x^4 + 84\,c\,\sigma_3^3\,(10 + c\,\sigma_3)\,x^3 + 48\,\sigma_3^2\,(160 + 80\,\sigma_1 - 80\,\sigma_2 + 70\,\sigma_3 + 7\,\sigma_3^2)\,x^2\right)\,(4\,c^2 - x);$$
$$g_2[x\_] = -\frac{3\,\sigma_3^2}{2^{20}}\left(80\,(1 + c)\,(2\,\sigma_1 - \sigma_2) + \sigma_3\,\left(10\,(11 + 11\,c + 7\,c^2) + 7\,(1 + c + c^2 + c^3)\,\sigma_3\right)\right)(1 - c)\,x^3;$$
$$g_3[x\_] = -\frac{3\,\sigma_3\,(1 - c)}{2^{18}}\left(80\,\sigma_2\,\left(4\,(c^2 + c - 1) - (1 + c)\,\sigma_3\right) + 80\,\sigma_1\,\left(4 + (1 + c)\,\sigma_3\right) + \sigma_3\,\left(70\,(1 + c)\,\sigma_3 - 80\,c\,(3 + 5\,c) + 7\,(1 + c)\,\sigma_3^2\right)\right)x^2;$$
$$g_0[x\_] = D[f[x], \{x, 3\}] - g_1[x] - g_2[x] - g_3[x]\ //\ Simplify;$$

$g_1[x] < 0$

In[116]:=
$$\text{CoefficientList}\left[21\,\sigma_3^4\,x^4 + 84\,c\,\sigma_3^3\,(10 + c\,\sigma_3)\,x^3 + 48\,\sigma_3^2\,(160 + 80\,\sigma_1 - 80\,\sigma_2 + 70\,\sigma_3 + 7\,\sigma_3^2)\,x^2,\ x\right]\ //\ N$$

Out[116]= $\{0.,\ 0.,\ 384.07,\ 0.518883\,(10. + 0.183483\,c)\,c,\ 0.0238016\}$

$g_2[x] \le 0$

In[117]:=
$$\text{CoefficientList}\left[80\,(1 + c)\,(2\,\sigma_1 - \sigma_2) + \sigma_3\,\left(10\,(11 + 11\,c + 7\,c^2) + 7\,(1 + c + c^2 + c^3)\,\sigma_3\right),\ c\right]\ //\ N$$

Out[117]= $\{238.68,\ 238.68,\ 13.0795,\ 0.235663\}$

$g_3[x] \le 0$

In[118]:=
$$\text{CoefficientList}\left[80\,\sigma_2\,\left(4\,(c^2 + c - 1) - (1 + c)\,\sigma_3\right) + 80\,\sigma_1\,\left(4 + (1 + c)\,\sigma_3\right) + \sigma_3\,\left(70\,(1 + c)\,\sigma_3 - 80\,c\,(3 + 5\,c) + 7\,(1 + c)\,\sigma_3^2\right),\ c\right]\ //\ N$$

Out[118]= $\{272.617,\ 326.527,\ 282.919\}$

$g_0[x] \le 0$

In[119]:= `Length[ CoefficientList[g0[x], x]]`

Out[119]= 3

In[120]:= `CoefficientList[g0[x], x][[2]] // N`

Out[120]= $-0.000188261 - 0.00660415\,c^2 + 0.00631164\,c^4$

In[121]:= `g0[0] // Simplify // N`

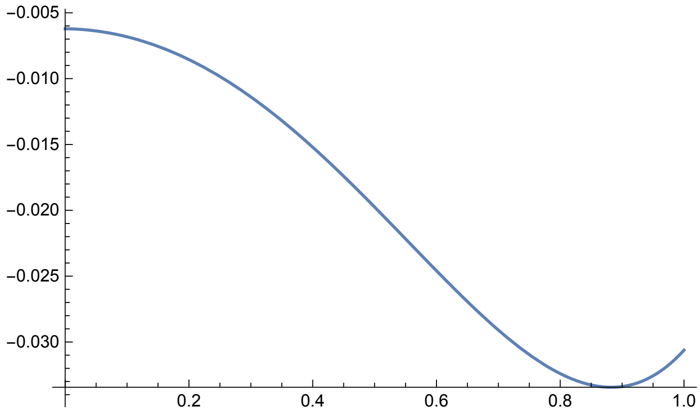
Out[121]= $-0.0234375\,c\,(0.265019 + 2.23389\,c^2 - 0.884121\,c^4)$

In[122]:= $g_0[4\,c^2]$ `// Simplify // N`

Out[122]= $-0.0234375\,c\,(0.265019 + 0.0321298\,c + 2.23389\,c^2 + 0.736328\,c^3 - 0.884121\,c^4 - 1.07719\,c^5)$

In[123]:= $\text{Plot}\left[\text{Evaluate}\left[\text{PolynomialQuotient}\left[g_0[4\,c^2],\ c,\ c\right]\right],\ \{c,\ 0,\ 1\}\right]$

Out[123]=



Function f is concave on $(4\,c^2, 8c)$

In[124]:= $g_1[x\_] = -\frac{1}{2^{25}}\left(7\,\sigma_3^4\,x^5 + 56\,c\,\sigma_3^3\,(6+\sigma_3)\,x^4 + 64\,\sigma_3^2\left(60 + 30\,\sigma_1 - 30\,\sigma_2 + 42\,\sigma_3 + 7\,\sigma_3^2\right)x^3 + 512\,c\,\sigma_3\left(7\left(\sigma_3+2\right)^3 + 80\left(\sigma_1 - 2\,\sigma_2\right) + 30\,(\sigma_1-\sigma_2)\,\sigma_3 - 56 + 36\,\sigma_3 + 20\,(4\,\sigma_2 - 5\,\sigma_3)\,c^2\right)x^2\right)(8\,c - x);$

$g_2[x\_] = -\frac{1}{2^{19}}\,\sigma_3^2\left(60\,\sigma_1 - 30\,\sigma_2 + \sigma_3\,(57 + 7\,\sigma_3)\right)\left(1-c^2\right)x^4;$

$g_3[x\_] = -\frac{1}{2^{13}}\left(7\,\sigma_3\left(\sigma_3+2\right)^3 + \left(30\,\sigma_1 - 30\,\sigma_2 - 73\right)\sigma_3^2 + 24\left(4 - 2\,\sigma_1 + \sigma_2\right)\sigma_2 + 8\left(7\,\sigma_1 - 10\,\sigma_2 - 25\right)\sigma_3 + \left(24\,\sigma_2^2 + 96\,\sigma_3 - 48\,\sigma_1\,\sigma_3 + 56\,\sigma_2\,\sigma_3 - 91\,\sigma_3^2\right)c^2\right)\left(1-c^2\right)x^2;$

```
g0[x_] = D[f[x], {x, 2}] - g1[x] - g2[x] - g3[x] // Simplify;
```

$g_1[x] < 0$

In[128]:= `g1[x] // N`

Out[128]= $-2.98023\times10^{-8}\,(8.\,c - 1.\,x)\left(93.9435\,c\left(3.43287 + 70.7297\,c^2\right)x^2 + 198.579\,x^3 + 2.139\,c\,x^4 + 0.00793387\,x^5\right)$

$g_2[x] < 0$

In[129]:= `g2[x] // N`

Out[129]= $-5.94241\times10^{-6}\left(1. - 1.\,c^2\right)x^4$

$g_3[x] \le 0$

In[130]:= `g3[x] // N`

Out[130]= $-0.00012207\left(1. - 1.\,c^2\right)\left(12.4095 + 38.8302\,c^2\right)x^2$

$g_0[x] \le 0$

In[131]:= `Length[CoefficientList[g0[x], x]]`

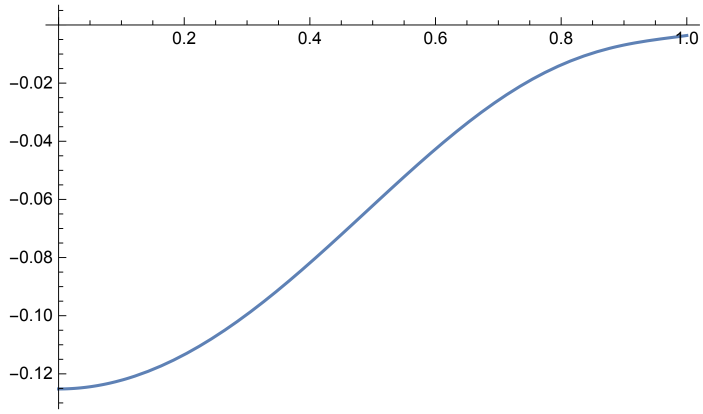
Out[131]= 3

In[132]:= `CoefficientList[g0[x], x][[2]] // N`

Out[132]= $-0.00621138\,c - 0.0523567\,c^3 + 0.0207216\,c^5$

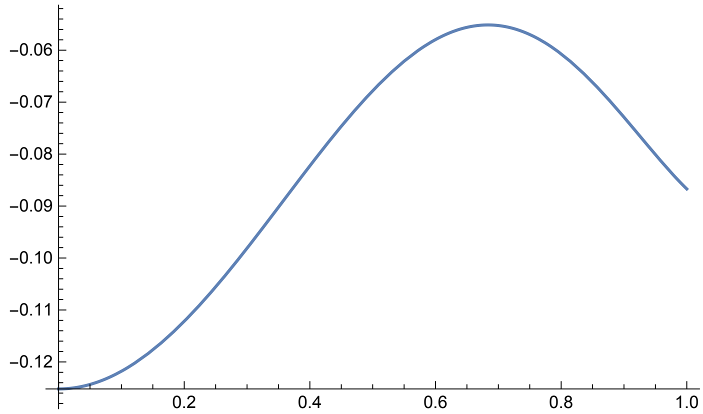
In[133]:= $\texttt{Plot}\left[g_0\left[4\,c^2\right], \{c, 0, 1\}\right]$

Out[133]=



In[134]:= `Plot[g0[8 c], {c, 0, 1}]`

Out[134]=



Function f is decreasing on (8c,4 $\left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right)$)

In[135]:= $g[t\_] = f'\left[4\left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}-2c\right)t+8c\right];$

In[136]:= `Length[CoefficientList[g[t], t]]`

Out[136]= 8

$g[t]=a_0+a_1 t+a_2 t^2+a_3 t^3+a_4 t^4+a_5 t^5+a_6 t^6+a_7 t^7<0$ if $a_0+\ldots+a_k<0$ for all k=0,...,7

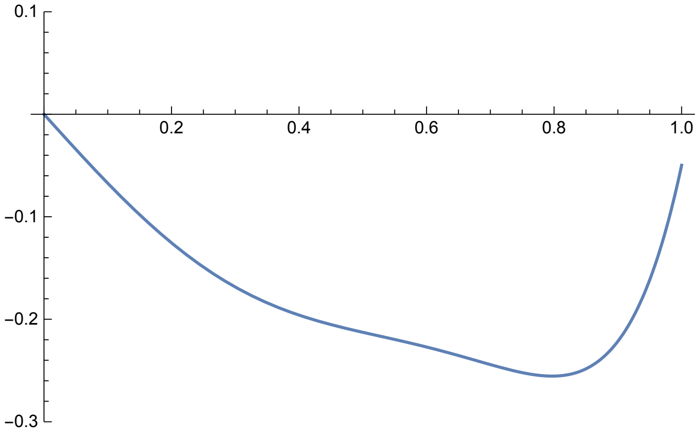
$a_0<0$

In[137]:= `g[0] /. c → 0 // Simplify`

Out[137]= 0

In[138]:= `Plot[Total[CoefficientList[g[t], t][[1 ;; 1]]], {c, 0, 1}, PlotRange → {-0.3, 0.1}]`

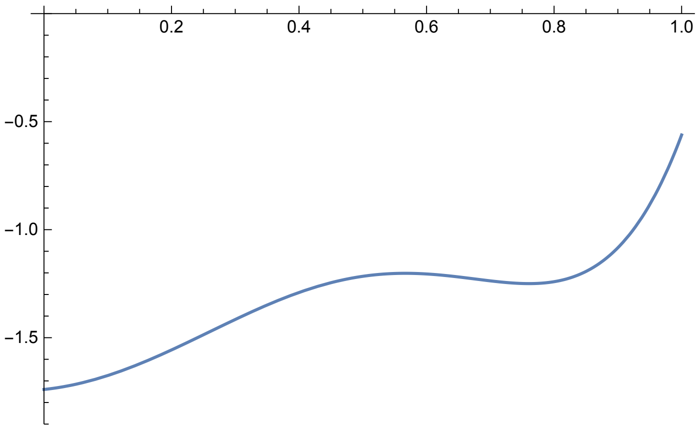
Out[138]=



$a_0 + a_1 < 0$

In[139]:= `Plot[Total[CoefficientList[g[t], t][[1 ;; 2]]], {c, 0, 1}, PlotRange → {-1.9, 0}]`

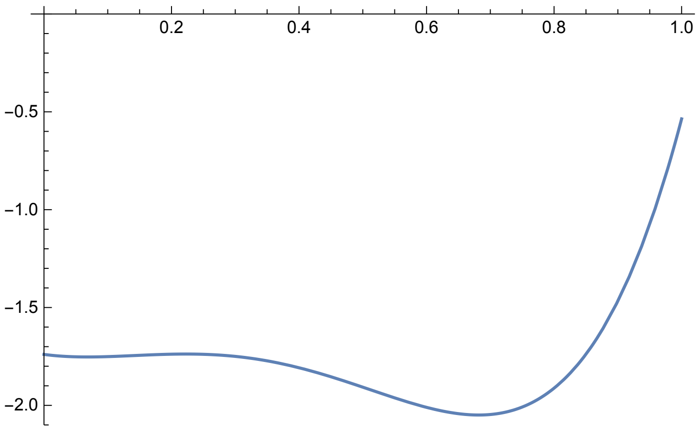
Out[139]=



$a_0 + a_1 + a_2 < 0$

In[140]:= `Plot[Total[CoefficientList[g[t], t][[1 ;; 3]]], {c, 0, 1}, PlotRange → {-2.1, 0}]`

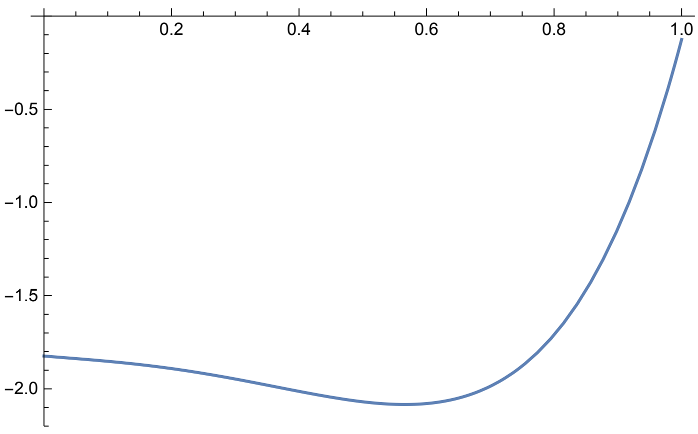
Out[140]=



$a_0 + a_1 + a_2 + a_3 < 0$

In[141]:= `Plot[Total[CoefficientList[g[t], t][[1 ;; 4]]], {c, 0, 1}, PlotRange → {-2.2, 0}]`

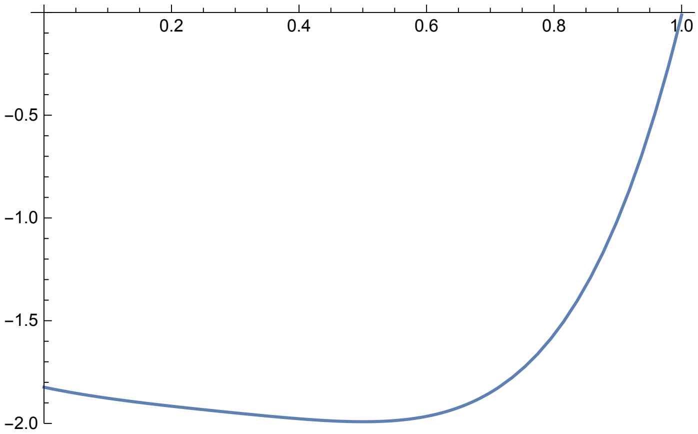
Out[141]=



$a_0 + a_1 + a_2 + a_3 + a_4 < 0$

In[142]:= `Plot[Total[CoefficientList[g[t], t][[1 ;; 5]]], {c, 0, 1}, PlotRange → {-2, 0}]`

Out[142]=



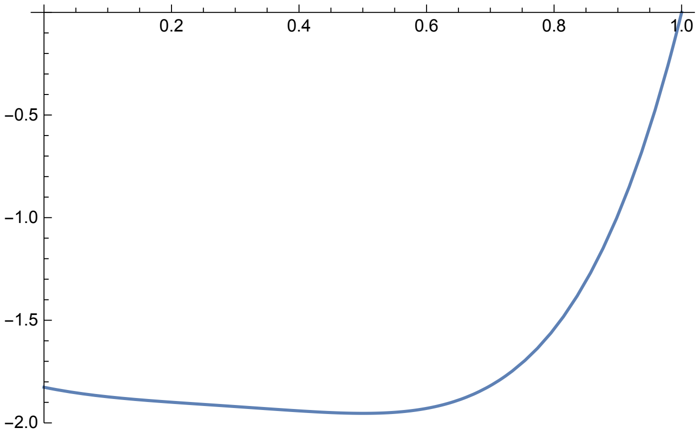
In[143]:= `Total[CoefficientList[g[t], t][[1 ;; 5]]] /. c → 1 // N`

Out[143]= -0.0120686

$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 < 0$

In[144]:= `Plot[Total[CoefficientList[g[t], t][[1 ;; 6]]], {c, 0, 1}, PlotRange → {-2, 0}]`

Out[144]=



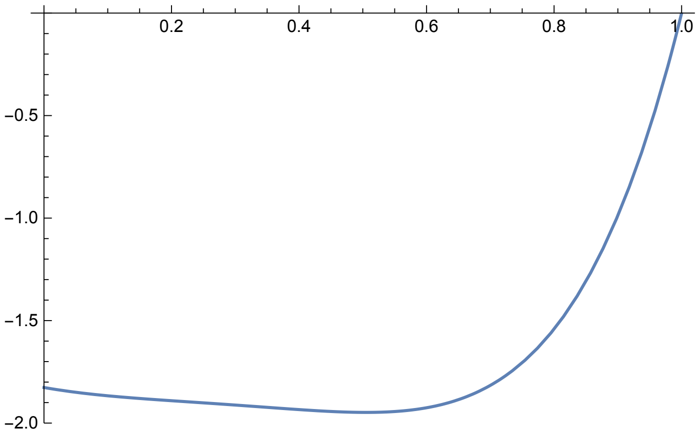
In[145]:= `Total[CoefficientList[g[t], t][[1 ;; 6]]] /. c → 1 // N`

Out[145]= -0.000520495

$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 0$

In[146]:=
```
Plot[Total[CoefficientList[g[t], t][[1 ;; 7]]], {c, 0, 1}, PlotRange → {-2, 0}]
```

Out[146]=



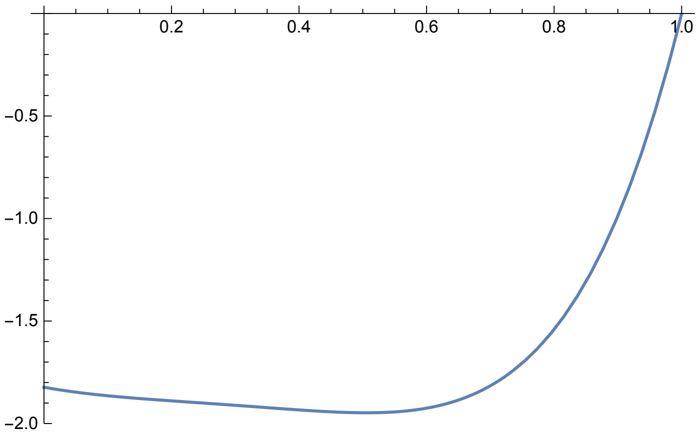
In[147]:=
```
Total[CoefficientList[g[t], t][[1 ;; 7]]] /. c → 1 // N
```

Out[147]= -8.35856×10^{-6}

$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 < 0$

In[148]:=
```
Plot[Total[CoefficientList[g[t], t][[1 ;; 8]]], {c, 0, 1}, PlotRange → {-2, 0}]
```

Out[148]=



In[149]:=
```
g[t] /. c → 1 /. t → 1 // FullSimplify
```

Out[149]= 0

Lemma 4

$x \in [-14, 0)$

In[150]:=

$$q_1[x\_] = -\frac{x}{32} + \frac{c}{8} + \frac{c^3}{8} - \frac{1}{512}\sigma_3\, c\, x^2;$$

$$q_2[x\_] = \frac{1}{4}\left(\left(\frac{1}{64}\sigma_3 x^2 + \frac{1}{4} c\, x - 1 + c^2\right)^2\right);$$

$$q_3[x\_] = \frac{1}{4}\left(-\frac{1}{16}\left(1 - \sigma_2 + 2\sigma_3\right) c^2 x^2 + \frac{1}{16}\left(2 - \sigma_1\right) x^2 - c\left(x - 8c\right)\right);$$

$q_1>0$

In[153]:= `D[q1[x], x] // N`

Out[153]= $-0.03125 - 0.000716732\, c\, x$

In[154]:= `D[q1[x], x] /. x → -14 // N`

Out[154]= $-0.03125 + 0.0100342\, c$

In[155]:= `q1[0] // Simplify`

Out[155]= $\frac{1}{8}\left(c + c^3\right)$

$q_3>0$

In[156]:= `D[q3[x], x] // Simplify // N`

Out[156]= $0.03125\left(-8.\, c + 0.0791323\, x - 0.253491\, c^2\, x\right)$

In[157]:= `q3[0] // Simplify`

Out[157]= $2\, c^2$

$x \in (4c(1+c^2), \left(6 - 3\sqrt{2} + \sqrt{68 - 46\sqrt{2}}\right)\left(1 + c^4\right)^2]$

$q_1<0$

In[158]:= `D[q1[x], x] // N`

Out[158]= $-0.03125 - 0.000716732\, c\, x$

In[159]:= `q1[8 c] // Simplify // N`

Out[159]= $0.125\left(-1.\, c + 0.816517\, c^3\right)$

$q_o<0$

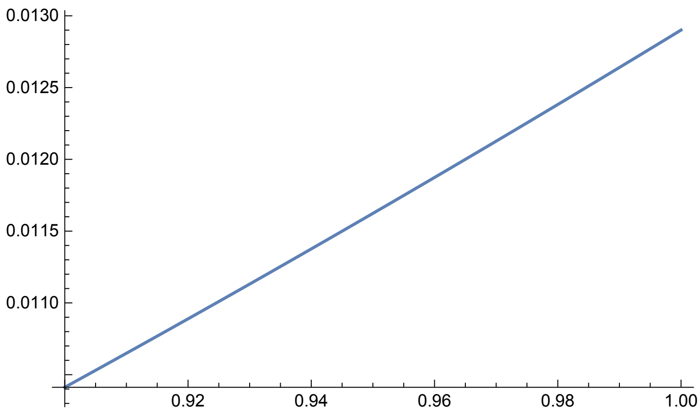
In[160]:= $q_0[x\_] = q_2[x] + q_3[x] - \frac{1}{2^{14}}\left(32\, c\, \sigma_3\, x^3 + \sigma_3^2\, x^4\right)$ `// Simplify;`

In[161]:= `Length[CoefficientList[q0[x], x]]`

Out[161]= 3

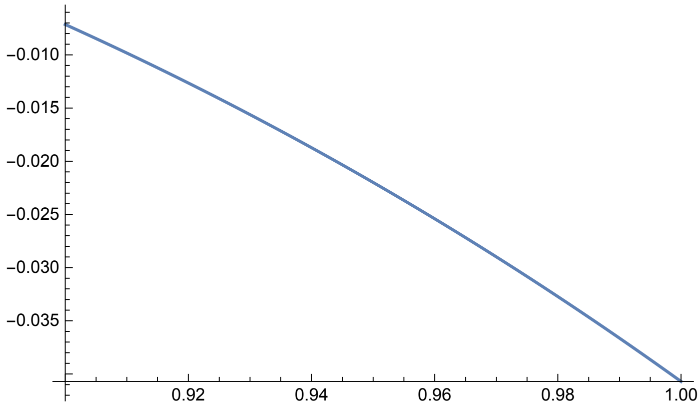
In[162]:= `Plot[Coefficient[q0[x], x, 2], {c, 0.9, 1}]`

Out[162]=



In[163]:=
```
Plot[Discriminant[q0[x], x], {c, 0.9, 1}]
```

Out[163]=



In[164]:=
```
CoefficientList[D[q0[x], x], c] // Simplify // N
```

Out[164]= {−0.000394045 x, −0.375, 0.0261953 x, 0.125}

In[165]:= Plot$\left[q_0\left[\left(6-3\sqrt{2}+\sqrt{68-46\sqrt{2}}\right)\left(1+c^4\right)^2\right], \{c, 0, 0.9\}, \text{PlotRange} \to \{0, 0.25\}\right]$

Out[165]=

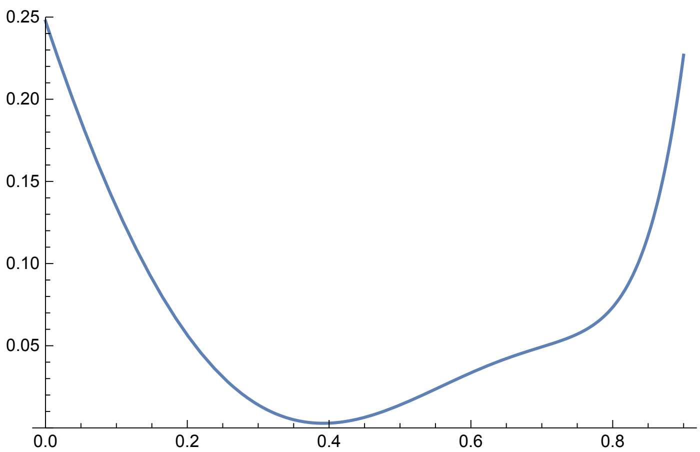


Table of best interpolants of arc c with corresponding errors

In[166]:= angles = $\left\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{8}, \frac{\pi}{12}\right\}$;

In[167]:=
```
sez4 = {{"φ", "α", "β", "γ", "error"}};
For[i = 1, i ≤ Length[angles], i++, φ0 = angles[[i]];
 x0 = Select[
    x /. NSolve[f[x] == 0 /. {c → Cos[φ0], s → Sin[φ0]}, x, Reals, WorkingPrecision → 30],
    # > -(1 - Cos[φ0])^2 && # < 0 &][[1]];
 y0 = y /. Solve[term6simpl == 0 /. {x → x0, c → Cos[φ0], s → Sin[φ0]}, y][[1]];
 β0 = β /. Solve[term5simpl == 0 /. {x → x0, y → y0, c → Cos[φ0], s → Sin[φ0]}, β][[1]];
 AppendTo[sez4, {φ0, NumberForm[1/8 (x0 + y0), {6, 5}], NumberForm[β0, {6, 5}],
    NumberForm[1/6 (y0 - x0 - 2 Cos[φ0]), {6, 5}], ScientificForm[
     N[e40[0, φ0, 1/8 (x0 + y0), β0, 1/6 (y0 - x0 - 2 Cos[φ0])]], 6]} /. φ → φ0]];
```

In[168]:=
```
Grid[sez4, Frame → All]
```

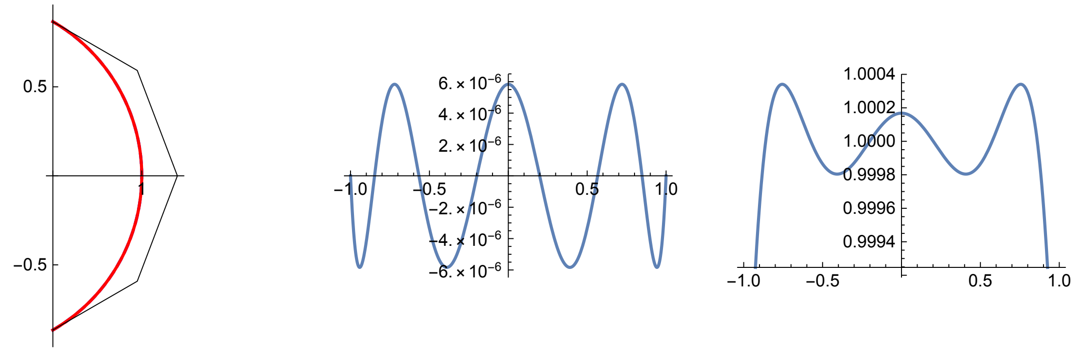
Out[168]=

φ	α	β	γ	error
$\frac{\pi}{2}$	0.87518	0.99857	1.49995	1.42325×10^{-4}
$\frac{\pi}{3}$	0.97471	0.59188	1.20039	5.8357×10^{-6}
$\frac{\pi}{4}$	0.99193	0.42228	1.10839	5.94378×10^{-7}
$\frac{\pi}{6}$	0.99840	0.27073	1.04680	2.34778×10^{-8}
$\frac{\pi}{8}$	0.99949	0.20014	1.02605	2.36051×10^{-9}
$\frac{\pi}{12}$	0.99990	0.13203	1.01149	9.23855×10^{-11}

You can draw the best interpolant of c, the corresponding error function and the curvature for arbitrary angle $\varphi 0 \in (0, \frac{\pi}{2}]$

In[169]:=
```
φ0 = π/3;
x0 = Select[
    x /. NSolve[f[x] == 0 /. {c → Cos[φ0], s → Sin[φ0]}, x, Reals, WorkingPrecision → 30],
    # > -(1 - Cos[φ0])^2 && # < 0 &][[1]];
y0 = y /. Solve[term6simpl == 0 /. {x → x0, c → Cos[φ0], s → Sin[φ0]}, y][[1]];
β0 = β /. Solve[term5simpl == 0 /. {x → x0, y → y0, c → Cos[φ0], s → Sin[φ0]}, β][[1]];
α0 = 1/8 (x0 + y0);
γ0 = 1/6 (y0 - x0 - 2 Cos[φ0]);
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[{b40[t, φ0, α0, β0, γ0][[1]], b40[t, φ0, α0, β0, γ0][[2]]},
     {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2, b3, b4}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3, b4}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[e40[t, φ0, α0, β0, γ0], {t, -1, 1}], Plot[Evaluate[
     (D[b40[t, φ0, α0, β0, γ0][[1]], t] × D[b40[t, φ0, α0, β0, γ0][[2]], {t, 2}] -
         D[b40[t, φ0, α0, β0, γ0][[2]], t] × D[b40[t, φ0, α0, β0, γ0][[1]], {t, 2}]) /
      (D[b40[t, φ0, α0, β0, γ0][[1]], t]^2 + D[b40[t, φ0, α0, β0, γ0][[2]], t]^2)^(3/2)],
    {t, -1, 1}]}] /. {φ → φ0, α → α0, β → β0, γ → γ0}
```

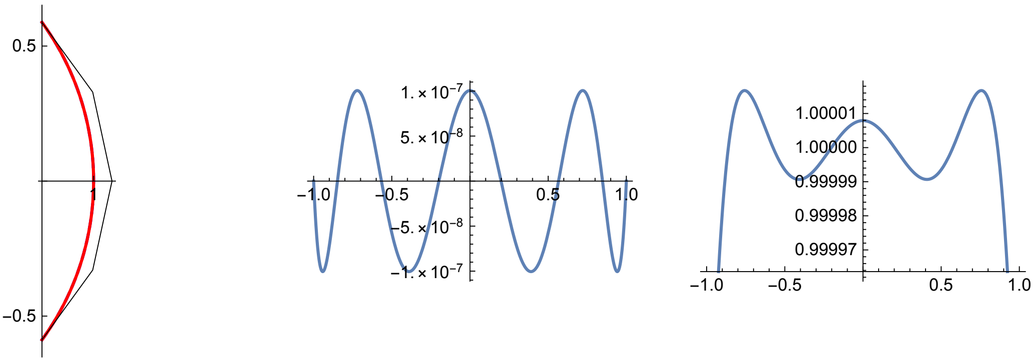
Out[175]=



If $\varphi0 < 0.6772$ there are two admissible solutions with alternating simplified radial error function

In[176]:=

```
φ0 = π/5;
x0 = {x /. FindRoot[f[x] /. {c → Cos[φ0], s → Sin[φ0]}, {x, -1/2 (1 - Cos[φ0])^2}],
   x /. FindRoot[f[x] /. {c → Cos[φ0], s → Sin[φ0]}, {x, -1}]};
ima = {{}, {}};
For[i = 1, i ≤ 2, i++,
 y0 = y /. Solve[term6simpl == 0 /. {x → x0[[i]], c → Cos[φ0], s → Sin[φ0]}, y][[1]];
 β0 =
  β /. Solve[term5simpl == 0 /. {x → x0[[i]], y → y0, c → Cos[φ0], s → Sin[φ0]}, β][[1]];
 α0 = 1/8 (x0[[i]] + y0);
 γ0 = 1/6 (y0 - x0[[i]] - 2 Cos[φ0]);
 ima[[i]] =
  {Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0}, PlotStyle → {Blue, Dashed},
      Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
     ParametricPlot[{b40[t, φ0, α0, β0, γ0][[1]], b40[t, φ0, α0, β0, γ0][[2]]},
      {t, -1, 1}, PlotStyle → Red],
     ListPlot[{b0, b1, b2, b3, b4}, PlotStyle → {PointSize[0.02], Black}],
     Graphics[{Black, Line[{b0, b1, b2, b3, b4}]}], AspectRatio → Automatic,
     PlotRange → All], Plot[e40[t, φ0, α0, β0, γ0], {t, -1, 1}], Plot[Evaluate[
      (D[b40[t, φ0, α0, β0, γ0][[1]], t] × D[b40[t, φ0, α0, β0, γ0][[2]], {t, 2}] -
          D[b40[t, φ0, α0, β0, γ0][[2]], t] × D[b40[t, φ0, α0, β0, γ0][[1]], {t, 2}]) /
       (D[b40[t, φ0, α0, β0, γ0][[1]], t]^2 + D[b40[t, φ0, α0, β0, γ0][[2]], t]^2)^(3/2)],
     {t, -1, 1}]} /. {φ → φ0, α → α0, β → β0, γ → γ0};]
GraphicsGrid[
 ima]
```



Out[180]=

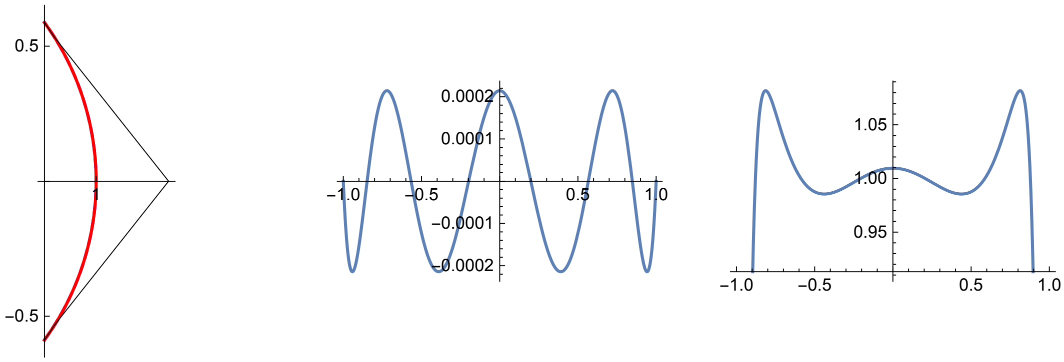


Table of best interpolants of arc d with corresponding errors

In[181]:=
```
angles = Reverse[{π/2, π/3, π/4, π/6, π/8, π/12}];
```

In[182]:=
```
sez4 = {{"φ", "α", "β", "γ", "error"}};
For[i = 1, i ≤ Length[angles], i++, φ0 = angles[[i]];
 x0 = Select[x /. NSolve[f[x] == 0 /. {c → Cos[φ0], s → Sin[φ0]},
      x, Reals, WorkingPrecision → 30], # > 0 && # < 14 &][[1]];
 y0 = y /. Solve[term6simpl == 0 /. {x → x0, c → Cos[φ0], s → Sin[φ0]}, y][[1]];
 β0 = β /. Solve[term5simpl == 0 /. {x → x0, y → y0, c → Cos[φ0], s → Sin[φ0]}, β][[1]];
 AppendTo[sez4, {2 π - 2 φ0, NumberForm[1/8 (x0 + y0), {6, 5}], NumberForm[β0, {6, 5}],
    NumberForm[1/6 (y0 - x0 - 2 Cos[φ0]), {6, 5}], ScientificForm[
     N[e40[0, φ0, 1/8 (x0 + y0), β0, 1/6 (y0 - x0 - 2 Cos[φ0])]], 6]} /. φ → φ0]];
```

In[183]:=
```
Grid[sez4, Frame → All]
```

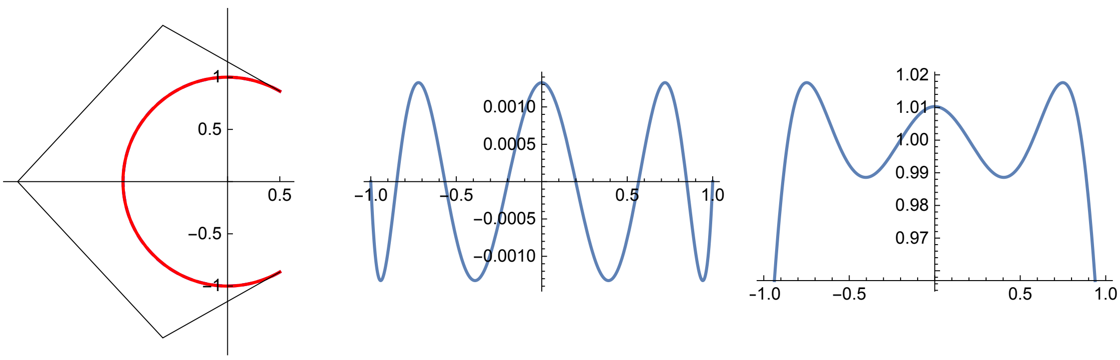
Out[183]=

φ	α	β	γ	error
$\frac{11\pi}{6}$	0.25762	2.33512	−3.35153	1.45985×10^{-2}
$\frac{7\pi}{4}$	0.06000	2.19627	−3.06840	1.03522×10^{-2}
$\frac{5\pi}{3}$	−0.11542	2.05518	−2.81103	7.19767×10^{-3}
$\frac{3\pi}{2}$	−0.40437	1.77231	−2.36754	3.25472×10^{-3}
$\frac{4\pi}{3}$	−0.61961	1.49705	−2.00895	1.32486×10^{-3}
π	−0.87518	0.99857	−1.49995	1.42325×10^{-4}

You can draw the best interpolant of d, the corresponding error function and the curvature for arbitrary angle $\varphi 0 \in [0, \frac{\pi}{2}]$

In[184]:=
```
φ0 = π/3;
x0 = Select[x /. NSolve[f[x] == 0 /. {c → Cos[φ0], s → Sin[φ0]},
      x, Reals, WorkingPrecision → 30], # > 0 && # < 14 &][[1]];
y0 = y /. Solve[term6simpl == 0 /. {x → x0, c → Cos[φ0], s → Sin[φ0]}, y][[1]];
β0 = β /. Solve[term5simpl == 0 /. {x → x0, y → y0, c → Cos[φ0], s → Sin[φ0]}, β][[1]];
α0 = 1/8 (x0 + y0);
γ0 = 1/6 (y0 - x0 - 2 Cos[φ0]);
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0 - 2 π},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[{b40[t, φ0, α0, β0, γ0][[1]], b40[t, φ0, α0, β0, γ0][[2]]},
     {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2, b3, b4}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3, b4}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[e40[t, φ0, α0, β0, γ0], {t, -1, 1}], Plot[Evaluate[
     -(D[b40[t, φ0, α0, β0, γ0][[1]], t] × D[b40[t, φ0, α0, β0, γ0][[2]], {t, 2}] -
         D[b40[t, φ0, α0, β0, γ0][[2]], t] × D[b40[t, φ0, α0, β0, γ0][[1]], {t, 2}]) /
      (D[b40[t, φ0, α0, β0, γ0][[1]], t]^2 + D[b40[t, φ0, α0, β0, γ0][[2]], t]^2)^(3/2)],
    {t, -1, 1}]}] /. {φ → φ0, α → α0, β → β0, γ → γ0}
```

Out[190]=



The best interpolant, the corresponding error function and the curvature for whole unite circle

In[191]:=
```
φ0 = 0;
x0 = 4 (6 - 3 √2 + √(68 - 46 √2));
y0 = y /. NSolve[term6simpl == 0 /. {x → x0, c → Cos[φ0], s → Sin[φ0]}, y][[1]];
β0 = β /. NSolve[ψ[u1] == 0 /. {α → (x0 + y0)/8, γ → 1/6 (-2 Cos[φ0] - x0 + y0),
      x → x0, y → y0, c → Cos[φ0], s → Sin[φ0]}, β][[1]];
α0 = 1/8 (x0 + y0);
γ0 = 1/6 (y0 - x0 - 2 Cos[φ0]);
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0 - 2 π}, PlotStyle →
      {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}], ParametricPlot[
     {b40[t, φ0, α0, β0, γ0][[1]], b40[t, φ0, α0, β0, γ0][[2]]}, {t, -1, 1}, PlotStyle →
      Red], ListPlot[{b0, b1, b2, b3, b4}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3, b4}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[e40[t, φ0, α0, β0, γ0], {t, -1, 1}], Plot[Evaluate[
     (D[b40[t, φ0, α0, β0, γ0][[1]], t] × D[b40[t, φ0, α0, β0, γ0][[2]], {t, 2}] -
        D[b40[t, φ0, α0, β0, γ0][[2]], t] × D[b40[t, φ0, α0, β0, γ0][[1]], {t, 2}]) /
      (D[b40[t, φ0, α0, β0, γ0][[1]], t]^2 + D[b40[t, φ0, α0, β0, γ0][[2]], t]^2)^(3/2)],
    {t, -1, 1}]}] /. {φ → φ0, α → α0, β → β0, γ → γ0}
```

Out[196]=