Parabolic case

In[1]:=

$$B_0[u\_] = (1-u)^2;\ B_1[u\_] = 2\,(1-u)\,u;\ B_2[u\_] = u^2;$$
$$b_0 = \{\mathrm{Cos}[\varphi], -\mathrm{Sin}[\varphi]\};\ b_1 = \{d, 0\};$$
$$b_2 = \{\mathrm{Cos}[\varphi], \mathrm{Sin}[\varphi]\};$$
$$\mathrm{b20}[t\_, \varphi\_, d\_] = \mathrm{Sum}\left[b_j\, B_j\left[(t+1)/2\right], \{j, 0, 2\}\right];$$
$$\mathrm{e20}[t\_, \varphi\_, d\_] = \mathrm{b20}[t, \varphi, d][[1]]^2 + \mathrm{b20}[t, \varphi, d][[2]]^2 - 1\ //\ \mathrm{FullSimplify};$$

In[6]:=

$$\psi_2[t\_] := \mathrm{e20}[t, \varphi, d];$$
$$\phi_2[t\_] := \sqrt{\mathrm{e20}[t, \varphi, d] + 1} - 1;$$

Extrema of ϕ_2 (and ψ_2)

In[8]:=

$$\mathrm{extrema2} = \mathrm{Solve}[D[\psi_2[t], t] == 0, t]\ //\ \mathrm{Simplify}$$

Out[8]=

$$\left\{\{t \to 0\}, \left\{t \to -\frac{\sqrt{-3+2\,d^2+\mathrm{Cos}[2\,\varphi]}}{\sqrt{2}\,\sqrt{(d-\mathrm{Cos}[\varphi])^2}}\right\}, \left\{t \to \frac{\sqrt{-3+2\,d^2+\mathrm{Cos}[2\,\varphi]}}{\sqrt{2}\,\sqrt{(d-\mathrm{Cos}[\varphi])^2}}\right\}\right\}$$

In[9]:=

$$\mathrm{Assuming}\left[d > \mathrm{Cos}[\varphi],\ \left(t\ /.\ \mathrm{extrema2}[[3]]\right) - \frac{\sqrt{d^2+c^2-2}}{d-c}\ /.\ \{c \to \mathrm{Cos}[\varphi], s \to \mathrm{Sin}[\varphi]\}\ //\ \mathrm{Simplify}\right]$$

Out[9]= 0

Function f

In[10]:=

$$f[d\_] = d^4 - 8\,d^3 + 2\,(c^2+4\,c+6)\,d^2 - 8\,c\,(4-c)\,d + c^4 - 8\,c^3 + 12\,c^2 + 4;$$

In[11]:=

$$\phi_2[0] + \phi_2[t\ /.\ \mathrm{extrema2}[[3]]]\ //\ \mathrm{Simplify}$$

Out[11]=

$$-2 + \frac{1}{2}\sqrt{(d+\mathrm{Cos}[\varphi])^2} + \sqrt{\frac{(-1+d^2)\,\mathrm{Sin}[\varphi]^2}{(d-\mathrm{Cos}[\varphi])^2}}$$

In[12]:=

$$4\left(\left(2\,(d-\mathrm{Cos}[\varphi]) - \frac{1}{2}\,(d-\mathrm{Cos}[\varphi])\,(d+\mathrm{Cos}[\varphi])\right)^2 - (-1+d^2)\,\mathrm{Sin}[\varphi]^2\right) - f[d]\ /.\ \{c \to \mathrm{Cos}[\varphi], s \to \mathrm{Sin}[\varphi]\}\ //\ \mathrm{Simplify}$$

Out[12]= 0

Value ϕ_2 (0)

In[13]:=

$$\phi_2[0]\ //\ \mathrm{Simplify}$$

Out[13]=

$$\frac{1}{2}\left(-2 + \sqrt{(d+\mathrm{Cos}[\varphi])^2}\right)$$

Lemma 3.1

In[14]:=

$$f\left[3-3\,c+c^2\right] - \left(-(1-c)^4\,\left(22\,(1-c^2) + (1-c^4) + 8\,c\,(1+c^2)\right)\right)\ //\ \mathrm{Simplify}$$

Out[14]= 0

In[15]:=
```
f[-x - c + 2] - (4 (1 - c)^4 + x (8 (1 - c)^2 c + 4 (2 - 2 c - x)^2 + x ((2 - 2 c - x)^2 + 4 c^2))) //
  Simplify
```

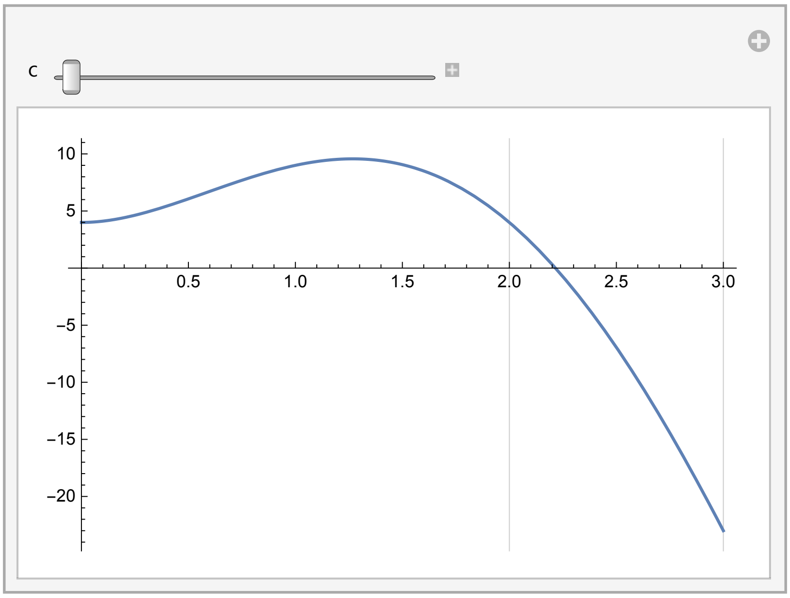
Out[15]= `0`

In[16]:=
```
f'[(1 - 2 c + c^2) x + 2 - c] -
   (-4 (1 - c)^2 (3 x^2 (1 - c)^2 c + 4 (1 - x c^2) + (4 - x^2 (1 - c)^3) x (1 - c) + 2 (c + x))) // Simplify
```

Out[16]= `0`

In[17]:=
```
Manipulate[Plot[d^4 - 8 d^3 + 2 (c^2 + 4 c + 6) d^2 - 8 c (4 - c) d + c^4 - 8 c^3 + 12 c^2 + 4,
   {d, 0, 3}, GridLines → {{2 - c, c^2 - 3 c + 3}, {}}], {c, 0, 1}]
```

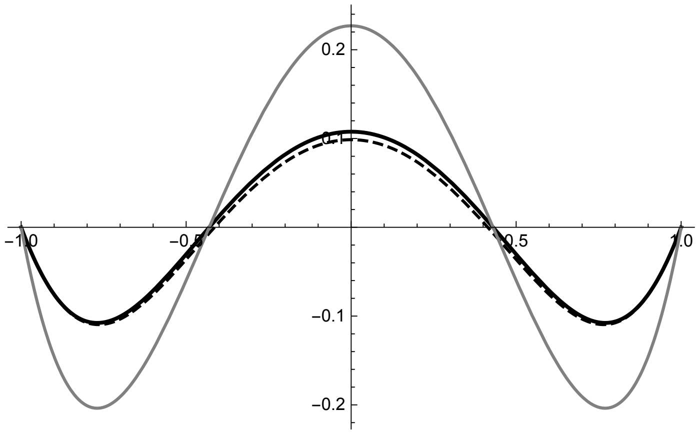
Out[17]=



Numerical computations

In[18]:=
```
dr = Select[d /. NSolve[f[d] == 0 /. c → Cos[π/2], d, Reals, WorkingPrecision → 20],
    # > 2 - Cos[π/2] && # < Cos[π/2]^2 - 3 Cos[π/2] + 3 &][[1]];
ds = Select[d /. NSolve[-4 + c^4 + 16 c d - 4 d^2 + d^4 - 2 c^2 (2 + 3 d^2) == 0 /. c → Cos[π/2],
      d, Reals, WorkingPrecision → 20], # ≥ 1 && # ≤ 1/2 (5 - 3 Cos[π/2]) &][[1]];
Plot[{Sqrt[e20[t, π/2, dr] + 1] - 1, Sqrt[e20[t, π/2, ds] + 1] - 1, e20[t, π/2, dr]},
 {t, -1, 1}, PlotStyle → {{Black, Thick}, {Black, Dashed}, Gray}]
```

Out[20]=



In[21]:=
```
angles = {π/2, π/3, π/4, π/6, π/8, π/12};
sez2 = {{"φ", "dr", "ψ2,0", "ds", "ϕ2,0"}};
For[i = 1, i ≤ Length[angles], i++, φ0 = angles[[i]];
 dr = Select[d /. NSolve[f[d] == 0 /. c → Cos[φ0], d, Reals, WorkingPrecision → 20],
    # > 2 - Cos[φ0] && # < Cos[φ0]^2 - 3 Cos[φ0] + 3 &][[1]];
 ds = Select[d /. NSolve[-4 + c^4 + 16 c d - 4 d^2 + d^4 - 2 c^2 (2 + 3 d^2) == 0 /. c → Cos[φ0],
      d, Reals, WorkingPrecision → 20], # ≥ 1 && # ≤ 1/2 (5 - 3 Cos[φ0]) &][[1]];
 AppendTo[sez2, {φ, NumberForm[dr, {6, 5}], ScientificForm[
      N[Sqrt[e20[0, φ0, dr] + 1] - 1], 6], NumberForm[ds, {6, 5}], ScientificForm[
      N[1 - Sqrt[e20[Sqrt[-3 + 2 ds^2 + Cos[2 φ0]] / (Sqrt[2] Sqrt[(ds - Cos[φ0])^2]), φ0, ds] + 1], 6]} /. φ → angles[[i]]]];
```

In[23]:=
```
Grid[sez2, Frame → All]
```

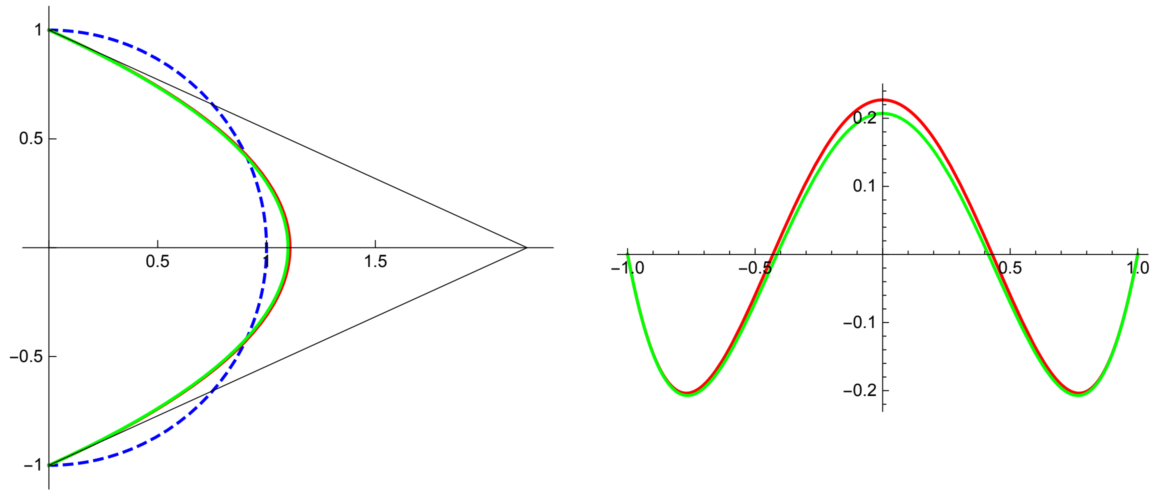
Out[23]=

φ	d_r	$\psi_{2,0}$	d_s	$\phi_{2,0}$
$\frac{\pi}{2}$	2.21535	1.07676×10^{-1}	2.19737	1.09554×10^{-1}
$\frac{\pi}{3}$	1.54728	2.36383×10^{-2}	1.54643	2.37668×10^{-2}
$\frac{\pi}{4}$	1.30843	7.76732×10^{-3}	1.30834	7.7828×10^{-3}
$\frac{\pi}{6}$	1.13713	1.57677×10^{-3}	1.13712	1.57746×10^{-3}
$\frac{\pi}{8}$	1.07713	5.03728×10^{-4}	1.07713	5.038×10^{-4}
$\frac{\pi}{12}$	1.03427	1.00191×10^{-4}	1.03427	1.00194×10^{-4}

The best interpolant for an arbitrary angle

In[24]:=
```
φ0 = π/2; b0 = {Cos[φ], -Sin[φ]}; b1 = {d, 0}; b2 = {Cos[φ], Sin[φ]};
dr = Select[d /. NSolve[f[d] == 0 /. c → Cos[φ0], d, Reals, WorkingPrecision → 20],
    # > 2 - Cos[φ0] && # < Cos[φ0]^2 - 3 Cos[φ0] + 3 &][[1]];
ds = Select[d /. NSolve[-4 + c^4 + 16 c d - 4 d^2 + d^4 - 2 c^2 (2 + 3 d^2) == 0 /. c → Cos[φ0],
      d, Reals, WorkingPrecision → 20], # ≥ 1 && # ≤ 1/2 (5 - 3 Cos[φ0]) &][[1]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
      PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
     ParametricPlot[b20[t, φ0, dr], {t, -1, 1}, PlotStyle → Red],
     ParametricPlot[b20[t, φ0, ds], {t, -1, 1}, PlotStyle → Green],
     ListPlot[{b0, b1, b2}, PlotStyle → {PointSize[0.02], Black}],
     Graphics[{Black, Line[{b0, b1, b2}]}], AspectRatio → Automatic, PlotRange → All],
    Plot[{e20[t, φ0, dr], e20[t, φ0, ds]}, {t, -1, 1},
     PlotStyle → {Red, Green}]}] /. {φ → φ0, d → ds}
```

Out[26]=



Parabolic case (revisited)

In[27]:=
```
B0[u_] = (1 - u)^2; B1[u_] = 2 (1 - u) u; B2[u_] = u^2;
b0 = {Cos[φ], -Sin[φ]}; b1 = {d, 0};
b2 = {Cos[φ], Sin[φ]};
b20[t_, φ_, d_] = Sum[bj Bj[(t + 1)/2], {j, 0, 2}];
ψ2[t_, d_] = b20[t, φ, d][[1]]^2 + b20[t, φ, d][[2]]^2 - 1 // FullSimplify;
ϕ2[t_, d_] = Sqrt[ψ2[t, d] + 1] - 1;
```

In[33]:=
```
q2[t_, d_] := 1/4 ((d - c)^2 t^2 - (d + c)^2 + 4);
```

In[34]:=
```
ψ2[t, d] - (t^2 - 1) q2[t, d] /. {c → Cos[φ]} // Simplify
```

Out[34]= 0

Lemma 5.1.

In[35]:=
```
q2[τ, 1] - 1/4 (1 - c) (3 + c + (1 - c) τ^2) // Simplify
```
$$q_2[\tau, 1] - \frac{1}{4}(1 - c)(3 + c + (1 - c)\tau^2) \text{ // Simplify}$$

Out[35]= 0

In[36]:=
$$\text{CoefficientList}[q_2[\tau, d], d][[-1]] - \left(-\frac{1}{4}(1 - \tau^2)\right) \text{ // Simplify}$$

Out[36]= 0

Lemma 5.2.

In[37]:=
$$q_2[0, d] - \frac{1}{4}\left(4 - (c + d)^2\right) \text{ // Simplify}$$

Out[37]= 0

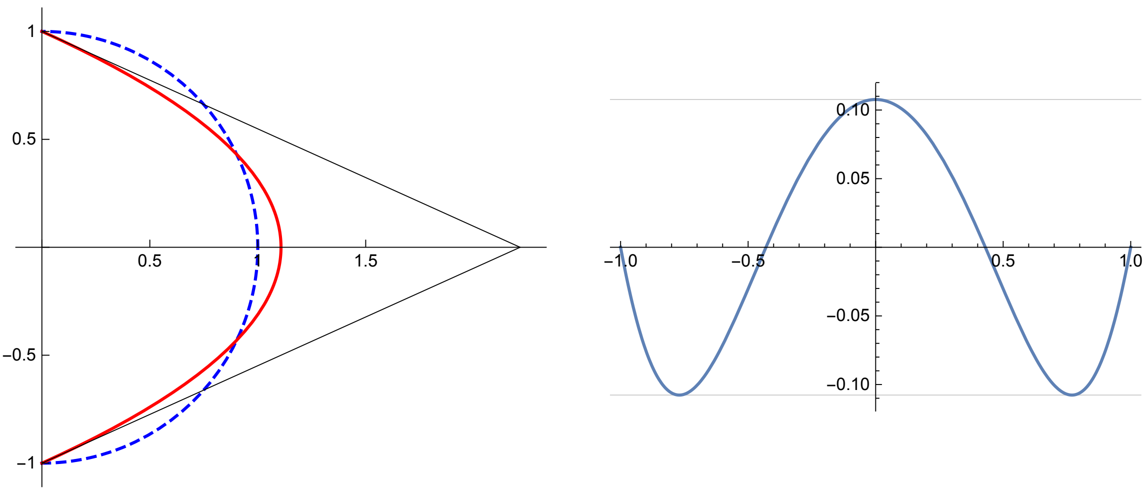
In[38]:=
$$q_2[1, d] - (1 - c\,d) \text{ // Simplify}$$

Out[38]= 0

The algorithm for the radial error

In[39]:=
```
φ0 = π/2; ϵ = 10^-5; b0 = {Cos[φ], -Sin[φ]};
b1 = {d, 0};
b2 = {Cos[φ], Sin[φ]};
τl = 0; τr = 1;
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 = Select[d /.
      NSolve[ψ2[τ0, d] == 0 /. {φ → φ0}, d, Reals, WorkingPrecision → 20], # > 1 &][[1]];
  If[ϕ2[0, d0] + ϕ2[Sqrt[d0^2 + Cos[φ0]^2 - 2]/(d0 - Cos[φ0]), d0] > 0 /. {φ → φ0}, τr = τ0, τl = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b20[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ2[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ2[0, d0], ϕ2[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
N[b20[t, φ0, d0]] // Simplify
```

Out[42]=

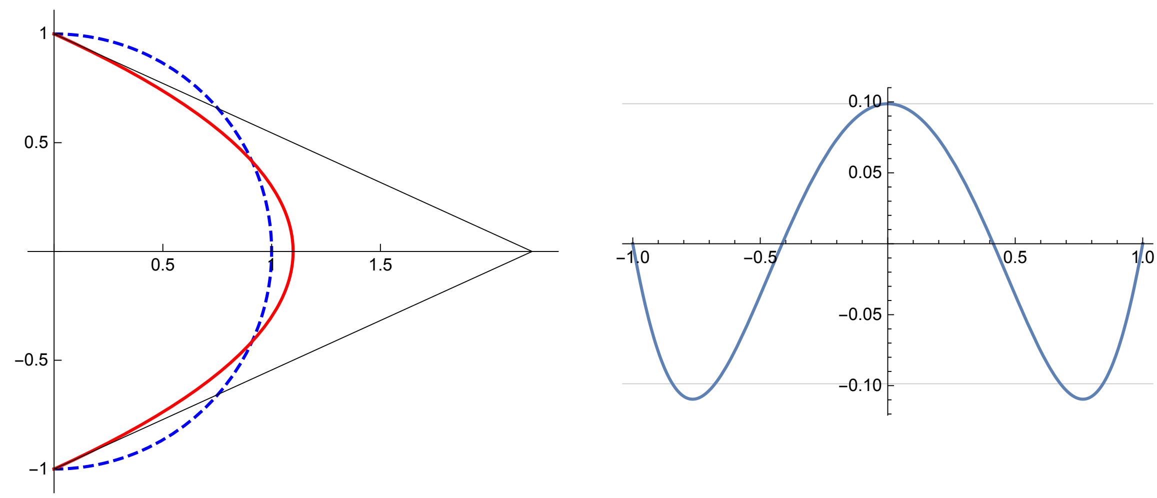


Out[43]= $\{1.10767 - 1.10767\, t^2,\ 0. + 1.\, t\}$

The algorithm for the simplified radial error

In[44]:=
```
φ0 = π/2; ϵ = 10^-5; b0 = {Cos[φ], -Sin[φ]};
b1 = {d, 0};
b2 = {Cos[φ], Sin[φ]};
τl = 0; τr = 1;
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 = Select[d /.
      NSolve[ψ2[τ0, d] == 0 /. {φ → φ0}, d, Reals, WorkingPrecision → 20], # > 1 &][[1]];
  If[ψ2[0, d0] + ψ2[Sqrt[d0^2 + Cos[φ0]^2 - 2]/(d0 - Cos[φ0]), d0] > 0 /. {φ → φ0}, τr = τ0, τl = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b20[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ2[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ2[0, d0], ϕ2[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
N[b20[t, φ0, d0]] // Simplify
```

Out[47]=



Out[48]=
```
{1.09868 - 1.09868 t^2, 0. + 1. t}
```

Cubic case

In[49]:=
```
B0[u_] = (1 - u)^3; B1[u_] = 3 (1 - u)^2 u; B2[u_] = 3 (1 - u) u^2; B3[u_] = u^3;
b0 = {Cos[φ], -Sin[φ]};
b1 = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b2 = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
b3 = {Cos[φ], Sin[φ]};
b31[t_, φ_, d_] = Sum[bj Bj[(t + 1)/2], {j, 0, 3}];
ψ3[t_, d_] = b31[t, φ, d][[1]]^2 + b31[t, φ, d][[2]]^2 - 1 // FullSimplify;
ϕ3[t_, d_] = Sqrt[ψ3[t, d] + 1] - 1;
```

In[54]:= $q_3[t\_, d\_] = \frac{1}{16}\left((3\,d\,c - 2\,s)^2\,t^2 + (3\,d\,s + 4\,c)^2 - 16\right);$

In[55]:= $\psi_3[t, d] - (t^2 - 1)^2\,q_3[t, d]$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[55]= 0

Lemma 6.1.

In[56]:= CoefficientList[q_3[t, d], d][[-1]] // Simplify

Out[56]= $\frac{9}{16}\left(s^2 + c^2\,t^2\right)$

In[57]:= $q_3[\tau, 0] - \left(-\frac{1}{4}\,s^2\,(4 - \tau^2)\right)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[57]= 0

Lemma 6.2.

In[58]:= $q_3[0, d] - \frac{1}{16}\left((3\,d\,s + 4\,c)^2 - 16\right)$ // Simplify

Out[58]= 0

In[59]:= $q_3[1, d] - \frac{3}{16}\left(3\,d^2 + 4\,s\,c\,d - 4\,s^2\right)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[59]= 0

The algorithm for the radial error

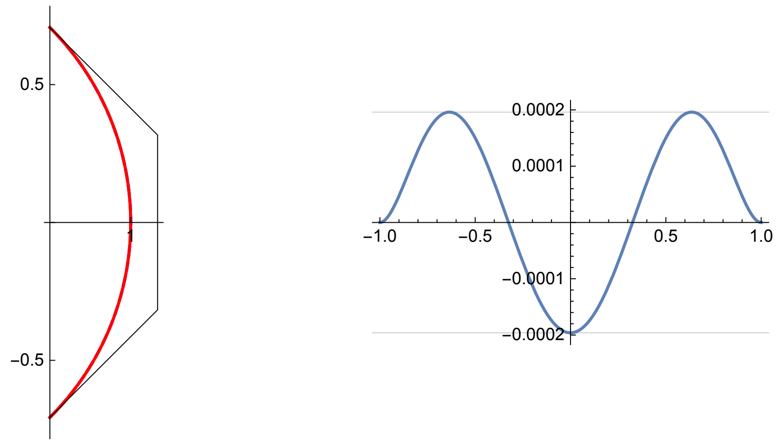
In[60]:= Solve[D[ψ_3[t, d], t] == 0, t] // Simplify

Out[60]= $\left\{\{t \to -1\}, \{t \to 0\}, \{t \to 1\}, \left\{t \to -\frac{\sqrt{12 - 3\,d^2 + 3\,(-4 + 3\,d^2)\,\mathrm{Cos}[2\,\varphi] - 20\,d\,\mathrm{Sin}[2\,\varphi]}}{\sqrt{2}\,\sqrt{(3\,d\,\mathrm{Cos}[\varphi] - 2\,\mathrm{Sin}[\varphi])^2}}\right\},\right.$

$\left.\left\{t \to \frac{\sqrt{12 - 3\,d^2 + 3\,(-4 + 3\,d^2)\,\mathrm{Cos}[2\,\varphi] - 20\,d\,\mathrm{Sin}[2\,\varphi]}}{\sqrt{2}\,\sqrt{(3\,d\,\mathrm{Cos}[\varphi] - 2\,\mathrm{Sin}[\varphi])^2}}\right\}\right\}$

In[61]:=
```
φ0 = π/4; ϵ = 10^-10;
b₀ = {Cos[φ], -Sin[φ]};
b₁ = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b₂ = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
b₃ = {Cos[φ], Sin[φ]};
τl = 0; τr = 1;
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 = Select[d /.
      NSolve[ψ₃[τ0, d] == 0 /. {φ → φ0}, d, Reals, WorkingPrecision → 20], # > 0 &][[1]];
  If[N[ϕ₃[0, d0] + ϕ₃[Sqrt[12 - 3 d0^2 + 3 (-4 + 3 d0^2) Cos[2 φ0] - 20 d0 Sin[2 φ0]] / (Sqrt[2] Sqrt[(3 d0 Cos[φ0] - 2 Sin[φ0])^2]), d0] /.
       {φ → φ0}, 20] > 0, τl = τ0, τr = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
      PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b31[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b₀, b₁, b₂, b₃}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b₀, b₁, b₂, b₃}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ₃[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ₃[0, d0], ϕ₃[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
N[b31[t, φ0, d0], 10] // Simplify
```

Out[65]=

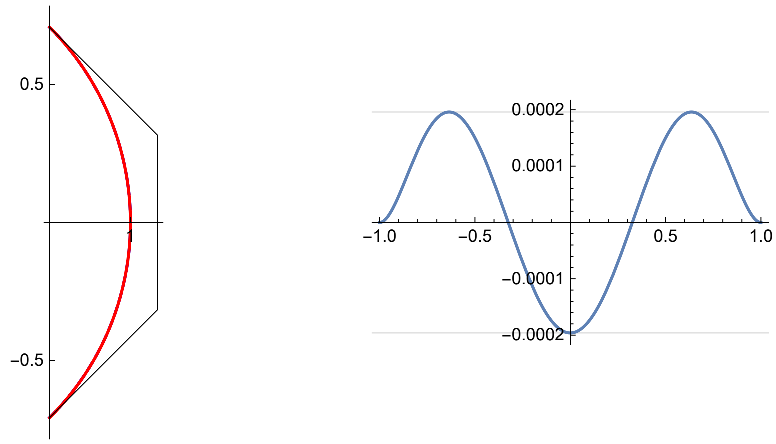


Out[66]= $\{0.9998039235 + 0. \times 10^{-10}\, t - 0.2926971423\, t^2 + 0. \times 10^{-10}\, t^3,\ 0. \times 10^{-12} + 0.767963029\, t + 0. \times 10^{-12}\, t^2 - 0.0608562482\, t^3\}$

The algorithm for the simplified radial error

In[67]:=
```
φ0 = π/4; ϵ = 10^-10;
b0 = {Cos[φ], -Sin[φ]};
b1 = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b2 = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
b3 = {Cos[φ], Sin[φ]};
τl = 0; τr = 1;
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 = Select[d /.
      NSolve[ψ3[τ0, d] == 0 /. {φ → φ0}, d, Reals, WorkingPrecision → 20], # > 0 &][[1]];
  If[N[ψ3[0, d0] + ψ3[Sqrt[12 - 3 d0^2 + 3 (-4 + 3 d0^2) Cos[2 φ0] - 20 d0 Sin[2 φ0]] / (Sqrt[2] Sqrt[(3 d0 Cos[φ0] - 2 Sin[φ0])^2]), d0] /.
       {φ → φ0}, 20] > 0, τl = τ0, τr = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b31[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2, b3}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ3[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ3[0, d0], ϕ3[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
N[b31[t, φ0, d0], 10] // Simplify
```

Out[71]=



Out[72]= $\{0.9998038950 + 0. \times 10^{-10}\, t - 0.2926971138\, t^2 + 0. \times 10^{-10}\, t^3,$
$0. \times 10^{-12} + 0.767963058\, t + 0. \times 10^{-12}\, t^2 - 0.0608562768\, t^3\}$

Quartic case

In[73]:=
```
B0[u_] = (1 - u)^4;
B1[u_] = 4 (1 - u)^3 u;
B2[u_] = 6 (1 - u)^2 u^2;
B3[u_] = 4 (1 - u) u^3;
B4[u_] = u^4;
b0 = {Cos[φ], -Sin[φ]};
b1 = {Cos[φ], -Sin[φ]} + Sqrt[3]/2 Sqrt[1 - Cos[φ] d] {Sin[φ], Cos[φ]};
b2 = {d, 0};
b3 = {Cos[φ], Sin[φ]} + Sqrt[3]/2 Sqrt[1 - Cos[φ] d] {Sin[φ], -Cos[φ]};
b4 = {Cos[φ], Sin[φ]};
b42[t_, φ_, d_] = Sum[bj Bj[(t + 1)/2], {j, 0, 4}]; // Simplify;
ψ4[t_, d_] = b42[t, φ, d][[1]]^2 + b42[t, φ, d][[2]]^2 - 1 // Simplify;
ϕ4[t_, d_] = Sqrt[ψ4[t, d] + 1] - 1;
```

In[79]:=
```
q4[t_, d_] =
  1/64 ((12 - 3 c^2 - 30 c d + 12 c^3 d + 9 d^2 + 12 Sqrt[3] c s Sqrt[1 - c d] - 12 Sqrt[3] s d Sqrt[1 - c d]) t^2 +
    52 - 13 c^2 - 12 c^3 d - 9 d^2 - 12 Sqrt[3] s d Sqrt[1 - c d] - 2 c (9 d + 10 Sqrt[3] s Sqrt[1 - c d]));
```

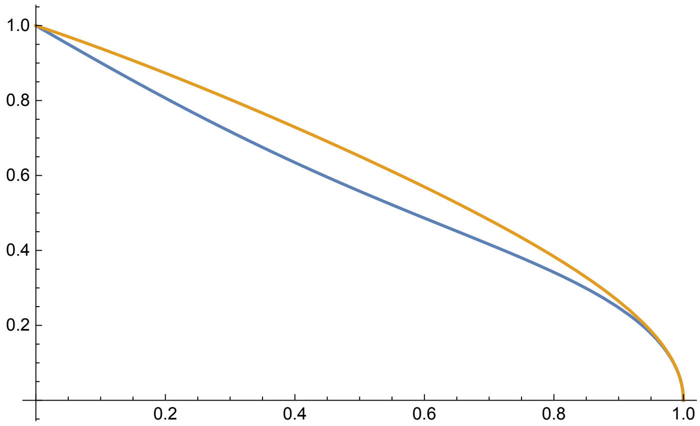
In[80]:=
```
ψ4[t, d] - (t^2 - 1)^3 q4[t, d] /. {c → Cos[φ], s → Sin[φ]} // Simplify
```

Out[80]= 0

In[81]:=
```
a4 = Sqrt[3]/3 s (c^2 + Sqrt[3] (1 - c)); b4 = Sqrt[3]/3 s (Sqrt[3 + c^2] - c);
```

In[82]:=
```
Plot[{a4 /. {s → Sqrt[1 - c^2]}, b4 /. {s → Sqrt[1 - c^2]}}, {c, 0, 1}]
```

Out[82]=



In[83]:=
```
Limit[(1 - a4^2)/c /. {s → Sqrt[1 - c^2]}, c → 0]
```

Out[83]= 2

In[84]:= $$\texttt{Limit}\left[\frac{1-\texttt{b4}^2}{\texttt{c}}\ \texttt{/.}\ \left\{\texttt{s}\to\sqrt{1-\texttt{c}^2}\right\},\ \texttt{c}\to 0\right]$$

Out[84]= $$\frac{2}{\sqrt{3}}$$

In[85]:= $$\texttt{r}_4[\texttt{t\_},\texttt{x\_}] = \texttt{Assuming}\left[\texttt{x} > 0,\ \texttt{q}_4\left[\texttt{t},\ \frac{1-\texttt{x}^2}{\texttt{c}}\right]\ \texttt{// Simplify}\right];$$

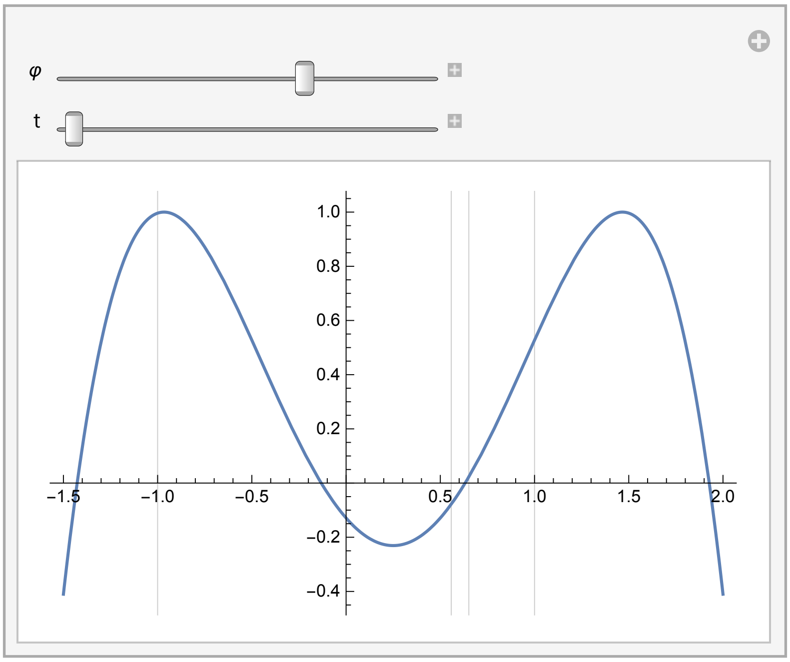
Lemma 7.1.

In[86]:= $$\texttt{CoefficientList}[\texttt{r}_4[\texttt{t},\texttt{x}],\texttt{x}][[-1]] - \left(-\frac{9}{64\,\texttt{c}^2}\left(1-\texttt{t}^2\right)\right)\ \texttt{// Simplify}$$

Out[86]= 0

In[87]:=
```
Manipulate[Plot[r4[t, x] /. {c → Cos[φ], s → Sin[φ]}, {x, -1.5, 2},
   GridLines → {{-1, a4 /. {c → Cos[φ], s → Sin[φ]}, b4 /. {c → Cos[φ], s → Sin[φ]}, 1},
     {}}], {{φ, π/3}, 0, π/2}, {t, 0, 1}]
```

Out[87]=



In[88]:= $$\texttt{r}_4[\texttt{t},-1] - \frac{1}{64}\left(39+8\sqrt{3}\,\texttt{c}\,\texttt{s}+9\,\texttt{t}^2+12\sqrt{3}\,\texttt{c}\,\texttt{s}\left(1-\texttt{t}^2\right)+\left(1-\texttt{c}^2\right)\left(13+3\,\texttt{t}^2\right)\right)\ \texttt{// Simplify}$$

Out[88]= 0

In[89]:= $$\texttt{r}_4[\texttt{t},\texttt{b4}] - \frac{1}{4}\left(1-\texttt{c}^2\right)\left(\texttt{c}\sqrt{3+\texttt{c}^2}-1-\texttt{c}^2\right)^2\ \texttt{/.}\ \{\texttt{c}\to\texttt{Cos}[\varphi],\ \texttt{s}\to\texttt{Sin}[\varphi]\}\ \texttt{// Simplify}$$

Out[89]= 0

In[90]:= $$\texttt{r}_4[\texttt{t},1] - \frac{1}{64}\left(52+12\,\texttt{t}^2+4\sqrt{3}\,\texttt{c}\,\texttt{s}\left(-5+3\,\texttt{t}^2\right)-\texttt{c}^2\left(13+3\,\texttt{t}^2\right)\right)\ \texttt{// Simplify}$$

Out[90]= 0

In[91]:= $s_4[t\_] = -16 + 24\sqrt{3} + 20\left(-1+\sqrt{3}\right) c - 15 c^2 - 16\sqrt{3}\, c^2 + 34 c^3 + 8\sqrt{3}\, c^3 - 16 c^4 - 8\sqrt{3}\, c^4 + 2 c^5 + 4\sqrt{3}\, c^5 - c^6 + \left(24\left(-2+\sqrt{3}\right) + 12\left(-5+3\sqrt{3}\right) c - \left(-15+8\sqrt{3}\right) c^2 - \left(-54+32\sqrt{3}\right) c^3 - 16\left(-2+\sqrt{3}\right) c^4 - \left(-6+4\sqrt{3}\right) c^5 + c^6\right) t^2;$

In[92]:= $r_4[t, a4] - \left(-\frac{1}{64}(1-c)^3 (1+c)\, s_4[t]\right)$ /. $s \to \sqrt{1-c^2}$ // Simplify

Out[92]= 0

In[93]:= $s_4[0] - \left(\left(4 - 2\left(1+\sqrt{3}\right) c^2\right)^2 + 12\left(2+\sqrt{3}\right) c^3 (1-c) + 20\left(\sqrt{3}-1\right) c \left(1-c^3\right) + \left(4\left(7-4\sqrt{3}\right) + c^2\right)\left(1-c^4\right) + 20\left(2\sqrt{3}-3\right) + 2\left(5-2\sqrt{3}\right) c^3 + 2\left(1+2\sqrt{3}\right) c^5\right)$ // Simplify

Out[93]= 0

In[94]:= $s_4[1] - 2\left(\left(7-4\sqrt{3}\right)\left(64+40\sqrt{3}-7c\right) + c\left(3-2\sqrt{3}\, c\right)^2 + c^3 + c^3\left(3\sqrt{3}-2-2c\right)^2\right)$ // Simplify

Out[94]= 0

Lemma 7.2.

c=0

In[95]:= $q_4[t, d]$ /. {c → 0, s → 1} // Simplify

Out[95]= $\frac{1}{64}\left(52 + 12 t^2 + 9 d^2 \left(-1+t^2\right) - 12\sqrt{3}\, d \left(1+t^2\right)\right)$

In[96]:= $q_4\left[\tau, \frac{2}{\sqrt{3}}\right] q_4[\tau, 2] - \left(-\frac{1}{32}\left(6\sqrt{3} - 8 + 3\left(2-\sqrt{3}\right)\left(1-\tau^2\right)\right)\right)$ /. {c → 0, s → 1} // Simplify

Out[96]= 0

Lemma 7.3

c=0

In[97]:= CoefficientList[$q_4[t, d]$ /. {c → 0, s → 1}, d][[-1]] // Simplify

Out[97]= $\frac{9}{64}\left(-1+t^2\right)$

In[98]:= $q_4[\tau, 0]$ /. {c → 0, s → 1} // Simplify

Out[98]= $\frac{1}{16}\left(13 + 3\tau^2\right)$

In[99]:= $lc[d\_] = \frac{3}{64}\left(2-\sqrt{3}\, d\right)^2;$

In[100]:= CoefficientList[$q_4[t, d]$ /. {c → 0, s → 1}, t][[-1]] - lc[d] // Simplify

Out[100]= 0

In[101]:= `lc[0] // Simplify`

Out[101]= $\frac{3}{16}$

In[102]:= `lc[2] -` $\frac{6\left(2-\sqrt{3}\right)}{16}$ `// Simplify`

Out[102]= 0

0<c<1

In[103]:= `lc[x_] =` $\frac{1}{64\, c^2}\left(9\, x^4 + 12\sqrt{3}\, c\, s\, x^3 - 6\left(3 - 5\, c^2 + 2\, c^4\right) x^2 - 12\sqrt{3}\, c\, s^3\, x + 9\, s^4\right)$;

In[104]:= `CoefficientList[`r_4`[t, x], t][[-1]] - lc[x] /. {c → Cos[φ], s → Sin[φ]} // Simplify`

Out[104]= 0

In[105]:= `lc[b4] /. {c → Cos[φ], s → Sin[φ]} // Simplify`

Out[105]= 0

In[106]:= `lc[`$-\frac{\sqrt{3}}{3}\, s\left(\sqrt{3+c^2}+c\right)$`] /. {c → Cos[φ], s → Sin[φ]} // Simplify`

Out[106]= 0

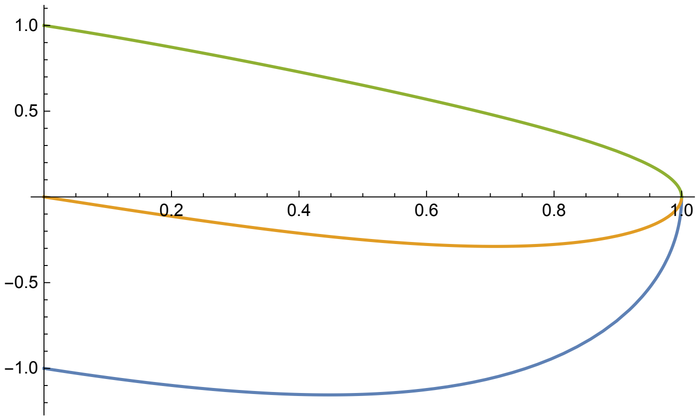
In[107]:= `D[lc[x], x] /. x →` $-\frac{c\, s}{\sqrt{3}}$ `/. {c → Cos[φ], s → Sin[φ]} // Simplify`

Out[107]= 0

Numbers $-\frac{\sqrt{3}}{3}\, s\left(\sqrt{3+c^2}+c\right)$, $-\frac{c\, s}{\sqrt{3}}$, b_4 are different for all c<1:

In[108]:= `Plot[{`$-\frac{\sqrt{3}}{3}\, s\left(\sqrt{3+c^2}+c\right)$ `/. s →` $\sqrt{1-c^2}$,
$-\frac{c\, s}{\sqrt{3}}$ `/. s →` $\sqrt{1-c^2}$`, b4 /. s →` $\sqrt{1-c^2}$`}, {c, 0, 1}]`

Out[108]=



In[109]:=
$$lc[1] - lc[a4] - \frac{1}{64}\left(c\left(4\sqrt{3}-9+4c^2\right)^2 + 12\sqrt{3}\,c\sqrt{1-c^2} + \left(24\sqrt{3}-36\right)(1-c) + 3\left(28\sqrt{3}-43\right)c\,(1-c) + \left(3+4\sqrt{3}\right)c^2(1-c) + \left(3+4\sqrt{3}\right)c^3(1-c) + 5\left(20\sqrt{3}-29\right)c^4(1-c) + \left(76\sqrt{3}-129\right)c^5(1-c) + \left(28\sqrt{3}-47\right)c^6(1-c) + \left(28\sqrt{3}-39\right)c^7(1-c) + \left(36\sqrt{3}-59\right)c^8(1-c) + \left(40\sqrt{3}-63\right)c^9(1-c) + 8\left(5\sqrt{3}-8\right)c^{10}\right) \;/.\; s \to \sqrt{1-c^2} \;//\; \text{Simplify}$$

Out[109]= 0

Lemma 7.4.

c=0

In[110]:=
$$q_4[0, d] - \frac{1}{64}\left(52 - 12\sqrt{3}\,d - 9d^2\right) \;/.\; \{c \to 0, s \to 1\} \;//\; \text{Simplify}$$

Out[110]= 0

In[111]:=
$$q_4[1, d] - \left(1 - \frac{1}{8}\left(3\sqrt{3}\,d\right)\right) \;/.\; \{c \to 0, s \to 1\} \;//\; \text{Simplify}$$

Out[111]= 0

c>0

In[112]:= `CoefficientList[r4[t, x], x][[-1]] // Simplify`

Out[112]=
$$\frac{9\left(-1+t^2\right)}{64\,c^2}$$

In[113]:= $f_0[x\_] = D[r_4[0, x], x];$

In[114]:= `Collect[f0[x], x]`

Out[114]=
$$\frac{-12\sqrt{3}\,c\,s - 20\sqrt{3}\,c^3 s}{64\,c^2} + \frac{\left(36+36c^2+24c^4\right)x}{64\,c^2} + \frac{9\sqrt{3}\,s\,x^2}{16\,c} - \frac{9x^3}{16\,c^2}$$

In[115]:= $f_1[x\_] = D[r_4[1, x], x];$

In[116]:= `Collect[f1[x], x]`

Out[116]=
$$\frac{-24\sqrt{3}\,c\,s - 8\sqrt{3}\,c^3 s}{64\,c^2} + \frac{3x}{2} + \frac{9\sqrt{3}\,s\,x^2}{8\,c}$$

In[117]:=
$$f_0[0] - \left(-\frac{\sqrt{3}\left(3+5c^2\right)s}{16\,c}\right) \;//\; \text{Simplify}$$

Out[117]= 0

In[118]:=
$$f_1[0] - \left(-\frac{\sqrt{3}\left(3+c^2\right)s}{8\,c}\right) \;//\; \text{Simplify}$$

Out[118]= 0

In[119]:= $f_0[1] - \left(\frac{9\,c + 6\,c^3 + \sqrt{3}\,s\,(6 - 5\,c^2)}{16\,c}\right)$ // Simplify

Out[119]= 0

In[120]:= $f_1[1] - \left(\frac{12\,c + \sqrt{3}\,s\,(6 - c^2)}{8\,c}\right)$ // Simplify

Out[120]= 0

In[121]:= $g_{01}[c\_] = 6\,(3 + \sqrt{3}) - 9\,(-1 + 2\sqrt{3})\,c;$
$g_{02}[c\_] = -5\sqrt{3}\,c^2 + (9 + \sqrt{3})\,c^5;$
$g_{03}[c\_] = c^3\,(2\,(3 + 8\sqrt{3}) - 2\,(9 + 4\sqrt{3})\,c + \sqrt{3}\,c^2\,(1 - c));$

In[124]:= $f_0[a4] - \frac{s\,(1 - c)}{16\,c}\,(g_{01}[c] + g_{02}[c] + g_{03}[c])$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[124]= 0

In[125]:= $g_{11}[c\_] = 6 - 4\,(3 - \sqrt{3})\,c;$
$g_{12}[c\_] = c^2\,(-13 + 6\sqrt{3} + 9\,c + (3 - 6\sqrt{3})\,c^2 + 3\,c^3);$

In[127]:= $f_1[a4] - \frac{\sqrt{3}\,s\,(1 - c)}{8\,c}\,(g_{11}[c] + g_{12}[c])$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[127]= 0

Algorithm for radial error

In[128]:= Solve[D[ψ_4[t, d], t] == 0, t] // Simplify

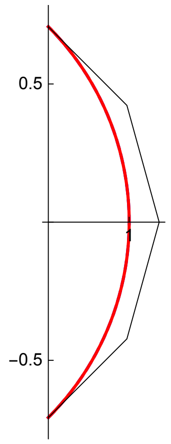
Out[128]= {{t → -1}, {t → -1}, {t → 0}, {t → 1}, {t → 1},

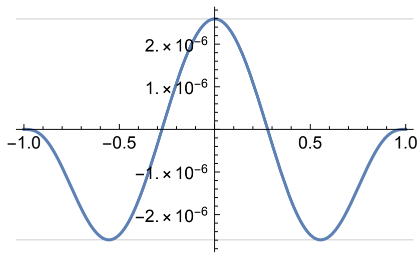
$\{t \to -\left(\left(\sqrt{\left(21 - 6\,d^2 - 10\,d\,\mathrm{Cos}[\varphi] - 3\,\mathrm{Cos}[2\,\varphi] - 2\,d\,\mathrm{Cos}[3\,\varphi] - 4\,d\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[\varphi] - 6\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[2\,\varphi]\right)}\right) \Big/ \left(\sqrt{\left(-7 - 6\,d^2 + 14\,d\,\mathrm{Cos}[\varphi] + \mathrm{Cos}[2\,\varphi] - 2\,d\,\mathrm{Cos}[3\,\varphi] + 8\,d\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[\varphi] - 4\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[2\,\varphi]\right)}\right)\right)\},$

$\{t \to \left(\sqrt{\left(21 - 6\,d^2 - 10\,d\,\mathrm{Cos}[\varphi] - 3\,\mathrm{Cos}[2\,\varphi] - 2\,d\,\mathrm{Cos}[3\,\varphi] - 4\,d\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[\varphi] - 6\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[2\,\varphi]\right)}\right) \Big/ \left(\sqrt{\left(-7 - 6\,d^2 + 14\,d\,\mathrm{Cos}[\varphi] + \mathrm{Cos}[2\,\varphi] - 2\,d\,\mathrm{Cos}[3\,\varphi] + 8\,d\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[\varphi] - 4\,\sqrt{3 - 3\,d\,\mathrm{Cos}[\varphi]}\,\mathrm{Sin}[2\,\varphi]\right)}\right)\}\}$

In[129]:=
```
φ0 = π/4; ϵ = 10^-20;
b₀ = {Cos[φ], -Sin[φ]};
b₁ = {Cos[φ], -Sin[φ]} + (√3/2) √(1 - Cos[φ] d) {Sin[φ], Cos[φ]};
b₂ = {d, 0};
b₃ = {Cos[φ], Sin[φ]} + (√3/2) √(1 - Cos[φ] d) {Sin[φ], -Cos[φ]};
b₄ = {Cos[φ], Sin[φ]};
τl = 0; τr = 1;
leftedge = (1 - b4^2)/c /. {c → Cos[φ0], s → Sin[φ0]};
rightedge = (1 - a4^2)/c /. {c → Cos[φ0], s → Sin[φ0]};
While[τr - τl > ϵ, τ0 = (1/2) (τr + τl);
  d0 = Select[d /. NSolve[ψ₄[τ0, d] == 0 /. {φ → φ0}, d, Reals],
     # > leftedge && # < rightedge &][[1]];
  If[N[ϕ₄[0, d0] + ϕ₄[(√(21 - 6 d0^2 - 10 d0 Cos[φ0] - 3 Cos[2 φ0] - 2 d0 Cos[3 φ0] -
            4 d0 √(3 - 3 d0 Cos[φ0]) Sin[φ0] - 6 √(3 - 3 d0 Cos[φ0]) Sin[2 φ0])) /
        (√(-7 - 6 d0^2 + 14 d0 Cos[φ0] + Cos[2 φ0] - 2 d0 Cos[3 φ0] +
            8 d0 √(3 - 3 d0 Cos[φ0]) Sin[φ0] - 4 √(3 - 3 d0 Cos[φ0]) Sin[2 φ0])),
         d0] /. {φ → φ0}, 20] > 0, τr = τ0, τl = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b42[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b₀, b₁, b₂, b₃, b₄}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b₀, b₁, b₂, b₃, b₄}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ₄[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ₄[0, d0], ϕ₄[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
b42[t, φ0, d0] // N // Simplify
```

Out[136]=





Out[137]= $\{1. - 0.30112\, t^2 + 0.00822386\, t^4,\ 0. + 0.775988\, t - 0.0688814\, t^3\}$

Quintic case

In[138]:=
```
B0[u_] = (1 - u)^5;
B1[u_] = 5 (1 - u)^4 u;
B2[u_] = 10 (1 - u)^3 u^2;
B3[u_] = 10 (1 - u)^2 u^3;
B4[u_] = 5 (1 - u) u^4;
B5[u_] = u^5;
b0 = {Cos[φ], -Sin[φ]}; b1 = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b2 = {(5 d (4 - 5 d^2) Cos[φ] + 4 (2 + 5 d^2) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), -(5 d ((4 - 5 d^2) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b3 = {(5 d (4 - 5 d^2) Cos[φ] + 4 (2 + 5 d^2) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), (5 d ((4 - 5 d^2) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b5 = {Cos[φ], Sin[φ]};
b4 = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
b53[t_, φ_, d_] = Sum[bj Bj[(t + 1)/2], {j, 0, 5}]; // Simplify;
ψ5[t_, d_] = b53[t, φ, d][[1]]^2 + b53[t, φ, d][[2]]^2 - 1 // Simplify;
ϕ5[t_, d_] = Sqrt[ψ5[t, d] + 1] - 1;
```

In[145]:=
```
q5[t_, d_] = 1/(1024 (5 d + 2 c s)^2) (-16 c^6 (36 - 16 t^2 + 25 d^2 (-1 + t^2)) +
    c^2 (-15625 d^6 (-1 + t^2) + 64 (-59 + 4 t^2) + 400 d^2 (69 + 5 t^2) + 2500 d^4 (-27 + 8 t^2)) +
    25 (64 + 625 d^6 t^2 - 300 d^4 (-3 + 2 t^2) + 16 d^2 (-34 + 9 t^2)) +
    8 c^4 (344 - 64 t^2 + 625 d^4 (1 + t^2) - 50 d^2 (36 + 13 t^2)) +
    20 c d (-200 d^2 (-11 + 2 t^2) + 16 c^4 (-3 + 2 t^2) + 625 d^4 (-3 + 2 t^2) -
       16 (19 + 6 t^2) - 8 c^2 (-44 - 8 t^2 + 25 d^2 (3 + 2 t^2))) s);
```

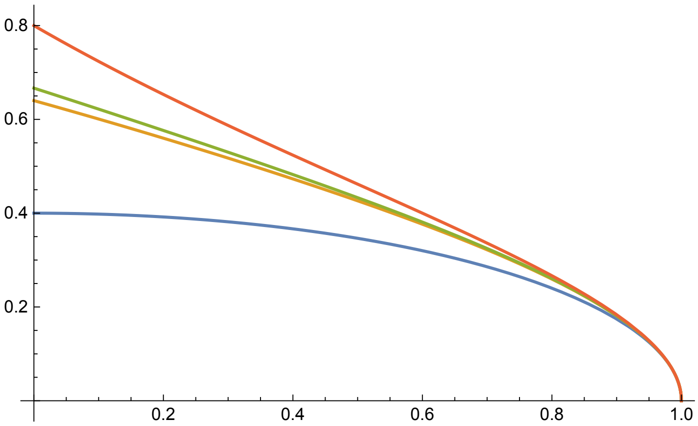
In[146]:=
```
ψ5[t, d] - (t^2 - 1)^4 q5[t, d] /. {c → Cos[φ], s → Sin[φ]} // Simplify
```

Out[146]= 0

In[147]:=
```
α5 = (2 s)/5; a5 = (16 s)/(25 + 15 c); b5 = (2 s)/(3 + 2 c); β5 = (4 s)/(5 (1 + c));
```

In[148]:=
```
Plot[{α5 /. {s → Sqrt[1 - c^2]}, a5 /. {s → Sqrt[1 - c^2]},
  b5 /. {s → Sqrt[1 - c^2]}, β5 /. {s → Sqrt[1 - c^2]}}, {c, 0, 1}]
```

Out[148]=



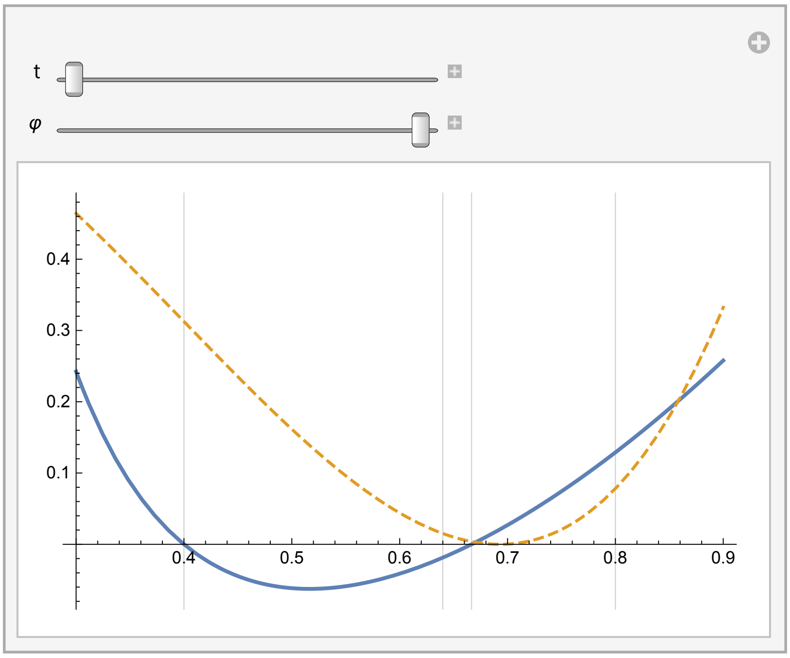
In[149]:=
```
lc[d_] = 1/(1024 (5 d + 2 c s)^2) (c^6 (256 - 400 d^2) + 25 d^2 (12 - 25 d^2)^2 +
    8 c^4 (-64 - 650 d^2 + 625 d^4) + c^2 (256 + 2000 d^2 + 20 000 d^4 - 15 625 d^6) +
    640 c^5 d s - 320 c^3 d (-4 + 25 d^2) s + 40 c d (-48 - 200 d^2 + 625 d^4) s);
```

In[150]:=
```
CoefficientList[q5[t, d], t][[-1]] - lc[d] // Simplify
```

Out[150]= 0

In[151]:=
```
Manipulate[
 Plot[{q5[t, d] /. {c → Cos[φ], s → Sin[φ]}, 5 lc[d] /. {c → Cos[φ], s → Sin[φ]}},
  {d, 0.3, 0.9}, GridLines → {{α5, a5, b5, β5} /. {c → Cos[φ], s → Sin[φ]}, {}},
  PlotStyle → {Thick, Dashed}], {t, 0, 1}, {{φ, π/2}, 0, π/2}]
```

Out[151]=



Lemma 8.1.

In[152]:=
```
r[d_] = (125 s d^3 + 100 c d^2 - 20 s (3 + c^2) d + 16 c s^2)/(32 (5 d + 2 s c));
```

In[153]:=
```
lc[d] - r[d]^2 /. {c → Cos[φ], s → Sin[φ]} // Simplify
```

Out[153]= 0

In[154]:=
```
r'[d] - 5 (125 s d^3 + 25 c (2 + 3 s^2) d^2 + 40 c^2 d s - 4 c s^2 (5 + c^2))/(16 (5 d + 2 c s)^2) // Simplify
```

Out[154]= 0

In[155]:=
```
r'[α5] - 5 (1 - c)/(8 (1 + c)^2) (1 + c + 2 c^2) /. {c → Cos[φ], s → Sin[φ]} // Simplify
```

Out[155]= 0

In[156]:= $$\texttt{r[b5]} - \left(-\frac{(1-c)^2\,(5-2c+2c^2)\,s}{4\,(3+2c)^2\,(5+3c+2c^2)}\right) \texttt{ /. \{c} \to \texttt{Cos}[\varphi]\texttt{, s} \to \texttt{Sin}[\varphi]\texttt{\} // Simplify}$$

Out[156]= 0

In[157]:= $$\texttt{r}[\beta 5] - \frac{(1-c)^2\,s}{4\,(1+c)\,(2+c+c^2)} \texttt{ /. \{c} \to \texttt{Cos}[\varphi]\texttt{, s} \to \texttt{Sin}[\varphi]\texttt{\} // Simplify}$$

Out[157]= 0

Lemma 8.2.

In[158]:= $$\texttt{g[x\_] = } q_5\left[t, \frac{4\,s}{5\,(1+c)}\,x\right];$$

In[159]:=
$$a_0[t\_] = (1+c)^5\,(75+15c^2+23c^4-c^6+(45c^2+22c^4+c^6)\,t^2);$$
$$a_1[t\_] = (1+c)^4\,(60c+136c^3-68c^5+75\,(1+c)+15c^2\,(1+c)+23c^4\,(1+c)-c^6\,(1+c)+ (-60c-112c^3-52c^5+45c^2\,(1+c)+22c^4\,(1+c)+c^6\,(1+c))\,t^2) \texttt{ // Simplify;}$$
$$a_2[t\_] = (1+c)^3\,(75+210c+366c^2+166c^3-474c^4-22c^5+2c^6-2c^7-c^8+ (-60c-159c^2-22c^3+147c^4-8c^5+19c^6+2c^7+c^8)\,t^2) \texttt{ // Simplify;}$$
$$a_3[t\_] = (1+c)^2\,(75+573c+576c^2-812c^3-308c^4+48c^5-20c^6-3c^8-c^9+ (-252c-219c^2+395c^3+125c^4-117c^5+11c^6+21c^7+3c^8+c^9)\,t^2);$$
$$a_4[t\_] = (-1+c^2)\,(-219-867c-624c^2-388c^3-500c^4-240c^5-28c^6-8c^7-5c^8- c^9+(96+348c-189c^2-365c^3+203c^4+195c^5+61c^6+29c^7+5c^8+c^9)\,t^2);$$
$$a_5[t\_] = (-1+c^2)\,(-219-291c-624c^2-964c^3-500c^4-240c^5-28c^6-8c^7-5c^8- c^9+(96-612c-189c^2+211c^3+203c^4+195c^5+61c^6+29c^7+5c^8+c^9)\,t^2);$$
$$a_6[t\_] = (-1+c)\,(-219-510c-1299c^2-1588c^3-1464c^4-740c^5-268c^6-36c^7-13c^8-6c^9- c^{10}+(-288-516c-417c^2+22c^3+414c^4+398c^5+256c^6+90c^7+34c^8+6c^9+c^{10})\,t^2);$$

In[166]:= $$\texttt{g''[x]} - \left(\frac{(1-c)^1}{2^3\,(1+c)^2\,(2x+c+c^2)^4}\right) (a_0[t]\,(1-x)+a_1[t]\,x\,(1-x)+a_2[t]\,x^2\,(1-x)+a_3[t]\,x^3\,(1-x)+ a_4[t]\,x^4\,(1-x)+a_5[t]\,x^5\,(1-x)+a_6[t]\,x^6) \texttt{ /. s} \to \texttt{Sqrt[1 - c\^{}2] // Simplify}$$

Out[166]= 0

All coefficients $a_j[t]$ are even polynomials of degree 2 and all $a_j[0]$ and $a_j[1]$ are polynomials in c:

In[167]:=
```
Grid[Table[{a_j[0], a_j[1]} // Simplify, {j, 0, 6}], Frame → All]
```

Out[167]=

$(1+c)^5\,(75+15c^2+23c^4-c^6)$	$15\,(1+c)^5\,(5+4c^2+3c^4)$
$-(1+c)^4\,(-75-135c-15c^2-151c^3-23c^4+45c^5+c^6+c^7)$	$(1+c)^4\,(75+75c+60c^2+84c^3+45c^4-75c^5)$
$(1+c)^3\,(75+210c+366c^2+166c^3-474c^4-22c^5+2c^6-2c^7-c^8)$	$3\,(1+c)^3\,(25+50c+69c^2+48c^3-109c^4-10c^5+7c^6)$
$-(1+c)^2\,(-75-573c-576c^2+812c^3+308c^4-48c^5+20c^6+3c^8+c^9)$	$3\,(1+c)^2\,(25+107c+119c^2-139c^3-61c^4-23c^5-3c^6+7c^7)$
$-(-1+c^2)\,(219+867c+624c^2+388c^3+500c^4+240c^5+28c^6+8c^7+5c^8+c^9)$	$3\,(-1+c^2)\,(-41-173c-271c^2-251c^3-99c^4-15c^5+11c^6+7c^7)$
$-(-1+c^2)\,(219+291c+624c^2+964c^3+500c^4+240c^5+28c^6+8c^7+5c^8+c^9)$	$3\,(-1+c^2)\,(-41-301c-271c^2-251c^3-99c^4-15c^5+11c^6+7c^7)$
$-(-1+c)\,(219+510c+1299c^2+1588c^3+1464c^4+740c^5+268c^6+36c^7+13c^8+6c^9+c^{10})$	$3\,(-1+c)\,(-169-342c-572c^2-522c^3-350c^4-114c^5-4c^6+18c^7+7c^8)$

(*) Polynomial $p(t)=c_0+c_1t+c_2t^2+\ldots+c_nt^n$ is nonnegative on [0,1] if $c_0+c_1+\ldots+c_k\geq0$ for all k=0,...,n.

By (*) for all j=0,...,6 we have $a_j[0]\geq0$:

In[168]:=
```
Grid[Table[Table[Total[CoefficientList[a_j[0], c][[1 ;; k]]], {k, 1, 12}], {j, 0, 6}],
 Frame → All]
```

Out[168]=

75	450	1215	2040	2588	2928	3232	3472	3577	3590	3585	3584
75	510	1515	2836	4168	5316	5892	5860	5693	5638	5633	5632
75	510	1731	3700	5032	4452	3132	2596	2573	2566	2561	2560
75	798	2595	3508	2152	772	540	548	525	518	513	512
219	1086	1491	1012	888	740	268	36	13	6	1	0
219	510	915	1588	1464	740	268	36	13	6	1	0
219	510	1299	1588	1464	740	268	36	13	6	1	0

By (*) for all j=0,...,3 we have $a_j[1]\geq0$:

In[169]:=
```
Grid[Table[Table[Total[CoefficientList[a_j[1], c][[1 ;; k]]], {k, 1, 10}], {j, 0, 6}],
 Frame → All]
```

Out[169]=

75	450	1260	2310	3330	4230	4980	5490	5715	5760
75	450	1260	2334	3450	4374	4740	4554	4299	4224
75	450	1332	2622	3498	3126	2220	1866	1899	1920
75	546	1620	2238	1578	726	396	330	363	384
123	642	1332	1566	1050	342	12	-54	-21	0
123	1026	1716	1566	1050	342	12	-54	-21	0
507	1026	1716	1566	1050	342	12	-54	-21	0

For j=4,5,6, a polynomial $a_j[1]$ is of the form $a_j[1]=(1-c)aa_j$ and by (*) for all j=4,5,6 we have $aa_j\geq0$:

In[170]:=
```
Grid[
 Table[Table[Total[CoefficientList[PolynomialQuotient[a_j[1], 1 - c, c], c][[1 ;; k]]],
   {k, 1, 9}], {j, 4, 6}], Frame → All]
```

Out[170]=

123	765	2097	3663	4713	5055	5067	5013	4992
123	1149	2865	4431	5481	5823	5835	5781	5760
507	1533	3249	4815	5865	6207	6219	6165	6144

Lemma 8.3.

In[171]:= $\psi_5[t, \alpha5] - \frac{(1-c)^5 (1-t^2)^4 t^2}{16 (1+c)}$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[171]= 0

In[172]:= $\psi_5[t, a5] - \left(-\frac{(1+c)(1-c)^4 (1-t^2)^4}{1024 (5+3c)^4 (8+5c+3c^2)^2} \left(48 (15925 + 4065 c + 1338 c^2 - 3846 c^3 - 1071 c^4 - 27 c^5) - 64 (1-c) (44 + 3c + 9c^2)^2 t^2\right)\right)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[172]= 0

In[173]:= $\psi_5[t, b5] - \frac{(1+c)(1-c)^4 (1-t^2)^4}{64 (3+2c)^4 (5+3c+2c^2)^2} \left(16 c (180 + 169 c + 123 c^2 + 27 c^3 + c^4) + 4 (1-c) (5 - 2c + 2c^2)^2 t^2\right)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[173]= 0

Lemma 8.4.

In[174]:= $f[d\_] = \frac{8 (5+3c^2) s + 20 c (9-c^2) d + 150 s d^2 - 125 c d^3}{32 (5d + 2sc)}$;

In[175]:= $q_5[0, d] - (f[d]^2 - 1)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[175]= 0

In[176]:= $g[d\_] = -4 (5 - 6c^2 + c^4) s + 60 s^2 c d + 75 s^3 d^2 - 125 c d^3$;

In[177]:= $f'[d] - \frac{5}{16 (5d + 2sc)^2} g[d]$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[177]= 0

In[178]:= $a5 - \left(\frac{2(1-c) s}{75 + 95c + 30c^2} x + \frac{16 s}{25 + 15 c}\right)$ /. x → 0 // Simplify

Out[178]= 0

In[179]:= $b5 - \left(\frac{2(1-c) s}{75 + 95c + 30c^2} x + \frac{16 s}{25 + 15 c}\right)$ /. x → 1 // Simplify

Out[179]= 0

In[180]:= $g\left[\frac{2(1-c)s}{75+95c+30c^2}x+\frac{16s}{25+15c}\right]-\frac{4s^3(1-c)^2}{(3+2c)^3(5+3c)^3}\big((3+2c)^{\wedge}3\,(335+297c+189c^{\wedge}2+27c^{\wedge}3)+30(3+2c)^{\wedge}2\,(8+5c+3c^{\wedge}2)x+3(5+2c+c^{\wedge}2)(3+2c)(1-3c)x^{\wedge}2-2(1-c)c\,x^{\wedge}3\big)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[180]= 0

In[181]:= h[d_] = q_5[1, d];

In[182]:= $h'[a5]-5\frac{4(1-c)^2s}{256(5+3c)^3(8+5c+3c^2)^3}(440863+1380116c+1950998c^2+1731140c^3+984672c^4+368604c^5+80946c^6+8748c^7+729c^8)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[182]= 0

In[183]:= $h''[b5]-\frac{375(1-c)}{32(3+2c)^2(5+3c+2c^2)^4}(2755+9325c+15649c^2+16453c^3+11136c^4+4638c^5+688c^6-356c^7-240c^8-48c^9)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[183]= 0

In[184]:= $h'''[b5]-\left(-\frac{375}{16(3+2c)(5+3c+2c^2)^5s}(650+8850c+24645c^2+41580c^3+53039c^4+52746c^5+40515c^6+22080c^7+7255c^8+216c^9-1016c^{10}-480c^{11}-80c^{12})\right)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[184]= 0

In[185]:= $h''''[d]-\frac{1875}{128}\left(1+\frac{320(5+2c^2+c^4)^2s^4}{(5d+2cs)^6}\right)$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[185]= 0

In[186]:= $r[\beta5]-r[a5]-\frac{(1-c)^2s(288+791c+915c^2+624c^3+260c^4+57c^5+9c^6)}{4(1+c)^2(5+3c)^2(2+c+c^2)(8+5c+3c^2)}$ /. {c → Cos[φ], s → Sin[φ]} // Simplify

Out[186]= 0

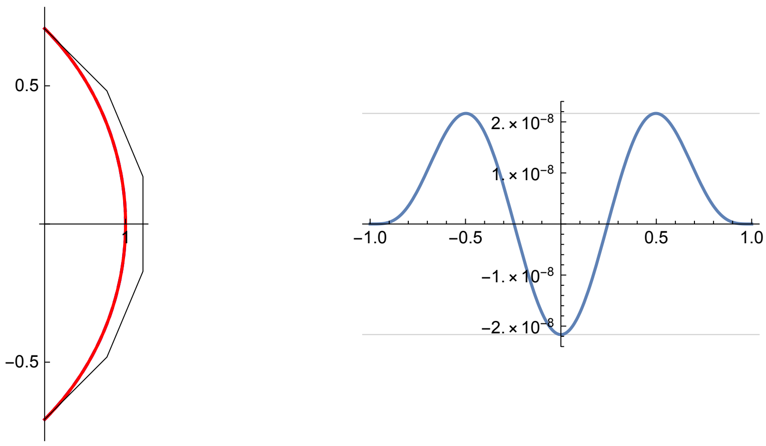
The algorithm for the radial error

In[187]:=
```
Solve[D[ψ5[t, d], t] == 0, t] // Simplify
```

Out[187]=
```
{{t → -1}, {t → -1}, {t → -1}, {t → 0}, {t → 1},
 {t → 1}, {t → 1}, {t → -((√(896 - 9130 d^2 - 13750 d^4 + 9375 d^6 +
        (-1248 + 11575 d^2 - 55000 d^4 + 15625 d^6) Cos[2 φ] + (384 - 2470 d^2 + 750 d^4) Cos[4 φ] -
        32 Cos[6 φ] + 25 d^2 Cos[6 φ] - 2072 d Sin[2 φ] + 32800 d^3 Sin[2 φ] -
        35000 d^5 Sin[2 φ] + 1120 d Sin[4 φ] - 2000 d^3 Sin[4 φ] - 56 d Sin[6 φ])) /
     (√2 √(-(-4 (1 + 25 d^2) Cos[φ] + 4 Cos[3 φ] + 5 d (14 - 25 d^2 + 2 Cos[2 φ]) Sin[φ])^2)))},
 {t → (√(896 - 9130 d^2 - 13750 d^4 + 9375 d^6 + (-1248 + 11575 d^2 - 55000 d^4 + 15625 d^6) Cos[2 φ] +
       (384 - 2470 d^2 + 750 d^4) Cos[4 φ] - 32 Cos[6 φ] + 25 d^2 Cos[6 φ] -
       2072 d Sin[2 φ] + 32800 d^3 Sin[2 φ] - 35000 d^5 Sin[2 φ] +
       1120 d Sin[4 φ] - 2000 d^3 Sin[4 φ] - 56 d Sin[6 φ])) /
    (√2 √(-(-4 (1 + 25 d^2) Cos[φ] + 4 Cos[3 φ] + 5 d (14 - 25 d^2 + 2 Cos[2 φ]) Sin[φ])^2))}}
```

In[188]:=
```
φ0 = π/4; ϵ = 10^-20;
b0 = {Cos[φ], -Sin[φ]}; b1 = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b2 = {(5 d (4 - 5 d^2) Cos[φ] + 4 (2 + 5 d^2) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), -(5 d ((4 - 5 d^2) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b3 = {(5 d (4 - 5 d^2) Cos[φ] + 4 (2 + 5 d^2) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), (5 d ((4 - 5 d^2) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b5 = {Cos[φ], Sin[φ]};
b4 = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
τl = 0; τr = 1;
leftedge = a5 /. {c → Cos[φ0], s → Sin[φ0]};
rightedge = b5 /. {c → Cos[φ0], s → Sin[φ0]};
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 = Select[d /. NSolve[ψ5[τ0, d] == 0 /. {φ → φ0}, d, Reals],
     # > leftedge && # < rightedge &][[1]];
  If[N[ϕ5[0, d0] + ϕ5[(√(896 - 9130 d0^2 - 13 750 d0^4 + 9375 d0^6 +
            (-1248 + 11 575 d0^2 - 55 000 d0^4 + 15 625 d0^6) Cos[2 φ0] +
            (384 - 2470 d0^2 + 750 d0^4) Cos[4 φ0] - 32 Cos[6 φ0] + 25 d0^2 Cos[6 φ0] -
            2072 d0 Sin[2 φ0] + 32 800 d0^3 Sin[2 φ0] - 35 000 d0^5 Sin[2 φ0] +
            1120 d0 Sin[4 φ0] - 2000 d0^3 Sin[4 φ0] - 56 d0 Sin[6 φ0])) /
         (√2 √(-(-4 (1 + 25 d0^2) Cos[φ0] + 4 Cos[3 φ0] + 5 d0 (14 - 25 d0^2 + 2 Cos[2 φ0])
                Sin[φ0])^2)), d0] /. {φ → φ0}, 20] > 0, τl = τ0, τr = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b53[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2, b3, b4, b5}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3, b4, b5}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ5[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ5[0, d0], ϕ5[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
b53[t, φ0, d0] // N // Simplify
```

Out[196]=



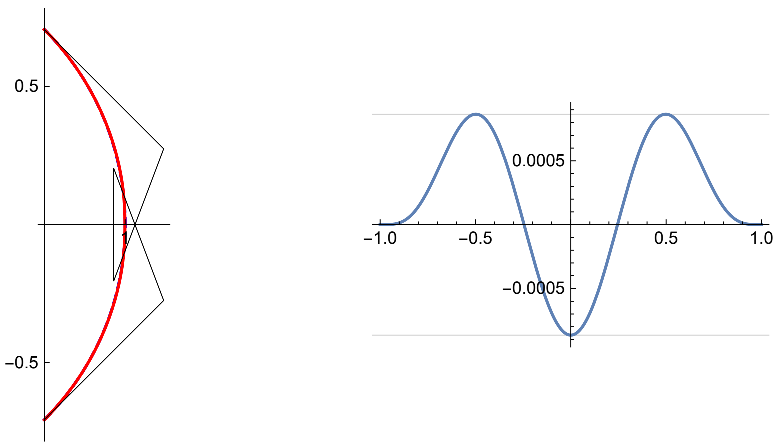
Out[197]= $\{1. - 0.303925\, t^2 + 0.0110313\, t^4,\ 2.77556 \times 10^{-17} + 0.779647\, t - 5.55112 \times 10^{-17}\, t^2 - 0.0733895\, t^3 + 0.000849009\, t^5\}$

The algorithm for the radial error for nonoptimal solutions

In[198]:=

```
φ0 = π/4; ϵ = 10^-20;
b0 = {Cos[φ], -Sin[φ]}; b1 = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b2 = {(5 d (4 - 5 d^2) Cos[φ] + 4 (2 + 5 d^2) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), -(5 d ((4 - 5 d^2) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b3 = {(5 d (4 - 5 d^2) Cos[φ] + 4 (2 + 5 d^2) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), (5 d ((4 - 5 d^2) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b5 = {Cos[φ], Sin[φ]};
b4 = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
τl = 0; τr = 1;
leftedge = a5 /. {c → Cos[φ0], s → Sin[φ0]};
rightedge = b5 /. {c → Cos[φ0], s → Sin[φ0]};
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 = Select[d /. NSolve[ψ5[τ0, d] == 0 /. {φ → φ0}, d, Reals], # > rightedge &][[1]];
  If[
   N[ϕ5[0, d0] + ϕ5[(√(896 - 9130 d0^2 - 13 750 d0^4 + 9375 d0^6 + (-1248 + 11 575 d0^2 - 55 000 d0^4 +
              15 625 d0^6) Cos[2 φ0] + (384 - 2470 d0^2 + 750 d0^4) Cos[4 φ0] - 32 Cos[6 φ0] +
           25 d0^2 Cos[6 φ0] - 2072 d0 Sin[2 φ0] + 32 800 d0^3 Sin[2 φ0] - 35 000 d0^5
            Sin[2 φ0] + 1120 d0 Sin[4 φ0] - 2000 d0^3 Sin[4 φ0] - 56 d0 Sin[6 φ0])) /
        (√2 √(-(-4 (1 + 25 d0^2) Cos[φ0] + 4 Cos[3 φ0] + 5 d0 (14 - 25 d0^2 + 2 Cos[2 φ0])
                Sin[φ0])^2)), d0] /. {φ → φ0}, 20] > 0, τl = τ0, τr = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b53[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b0, b1, b2, b3, b4, b5}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b0, b1, b2, b3, b4, b5}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ5[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ5[0, d0], ϕ5[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
b53[t, φ0, d0] // N // Simplify
```

Out[206]=

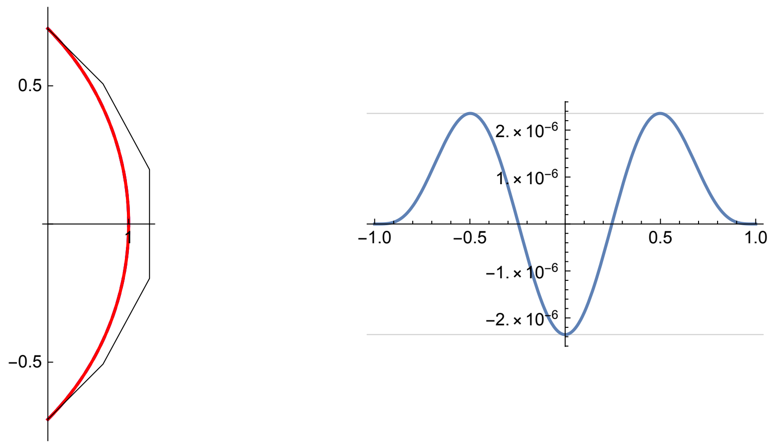


Out[207]=

$$\{0.999134 - 0.0435723\, t^2 - 0.248455\, t^4,\ 0. + 0.350433\, t - 5.55112 \times 10^{-17}\, t^2 + 0.526418\, t^3 - 0.169744\, t^5\}$$

In[208]:=
```
φ0 = π/4; ϵ = 10^-20;
b₀ = {Cos[φ], -Sin[φ]}; b₁ = {Cos[φ], -Sin[φ]} + d {Sin[φ], Cos[φ]};
b₂ = {(5 d (4 - 5 d²) Cos[φ] + 4 (2 + 5 d²) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), -(5 d ((4 - 5 d²) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b₃ = {(5 d (4 - 5 d²) Cos[φ] + 4 (2 + 5 d²) Sin[φ])/(4 (5 d + 2 Sin[φ] Cos[φ])), (5 d ((4 - 5 d²) Sin[φ] - 6 d Cos[φ]))/(4 (5 d + 2 Sin[φ] Cos[φ]))};
b₅ = {Cos[φ], Sin[φ]};
b₄ = {Cos[φ], Sin[φ]} + d {Sin[φ], -Cos[φ]};
τl = 0; τr = 1;
leftedge = a5 /. {c → Cos[φ0], s → Sin[φ0]};
rightedge = b5 /. {c → Cos[φ0], s → Sin[φ0]};
While[τr - τl > ϵ, τ0 = 1/2 (τr + τl);
  d0 =
   Select[d /. NSolve[ψ₅[τ0, d] == 0 /. {φ → φ0}, d, Reals], # > 0 && # < leftedge &][[1]];
  If[N[ϕ₅[0, d0] + ϕ₅[(√(896 - 9130 d0² - 13750 d0⁴ + 9375 d0⁶ +
            (-1248 + 11575 d0² - 55000 d0⁴ + 15625 d0⁶) Cos[2 φ0] +
            (384 - 2470 d0² + 750 d0⁴) Cos[4 φ0] - 32 Cos[6 φ0] + 25 d0² Cos[6 φ0] -
            2072 d0 Sin[2 φ0] + 32800 d0³ Sin[2 φ0] - 35000 d0⁵ Sin[2 φ0] +
            1120 d0 Sin[4 φ0] - 2000 d0³ Sin[4 φ0] - 56 d0 Sin[6 φ0])) /
         (√2 √(-(-4 (1 + 25 d0²) Cos[φ0] + 4 Cos[3 φ0] + 5 d0 (14 - 25 d0² + 2 Cos[2 φ0])
                 Sin[φ0])²)), d0] /. {φ → φ0}, 20] > 0, τl = τ0, τr = τ0]];
GraphicsRow[{Show[ParametricPlot[{Cos[φ], Sin[φ]}, {φ, -φ0, φ0},
     PlotStyle → {Blue, Dashed}, Ticks → {{0, 0.5, 1, 1.5}, {-1, -0.5, 0, 0.5, 1}}],
    ParametricPlot[b53[t, φ0, d0], {t, -1, 1}, PlotStyle → Red],
    ListPlot[{b₀, b₁, b₂, b₃, b₄, b₅}, PlotStyle → {PointSize[0.02], Black}],
    Graphics[{Black, Line[{b₀, b₁, b₂, b₃, b₄, b₅}]}], AspectRatio → Automatic,
    PlotRange → All], Plot[ϕ₅[t, d0] /. {φ → φ0}, {t, -1, 1},
    GridLines → {{}, {-ϕ₅[0, d0], ϕ₅[0, d0]} /. {φ → φ0}}]}] /. {φ → φ0, d → d0}
b53[t, φ0, d0] // N // Simplify
```

Out[216]=



Out[217]=

$$\{0.999998 - 0.335621\, t^2 + 0.0427299\, t^4,\ -2.77556\times 10^{-17} + 0.819352\, t - 0.121097\, t^3 - 2.77556\times 10^{-17}\, t^4 + 0.00885246\, t^5\}$$