MAPS FROM THE MINIMAL GROPE TO AN ARBITRARY GROPE

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Abstract. We give a systematic definition of the fundamental groups of gropes, which we call grope groups. We show that there exists a nontrivial homomorphism from the minimal grope group M to another grope group G only if G is the free product of M with another grope group.

1. Introduction

Here we study groups whose classifying spaces are (open infinite) gropes (a recent short note on gropes in general is [10]). In algebra these groups first appeared in the proof of a lemma by Alex Heller [7] as follows. Let φ_0 be a homomorphism from the free group F_0 on one generator α to any perfect group P. Let

$$\varphi_0(\alpha) = [p_0, p_1][p_2, p_3] \cdots [p_{2n-2}, p_{2n-1}] \in P \qquad (*)$$

then we can extend φ_0 to a homomorphism φ_1 of a (nonabelian) free group F_1 on $2n$ generators $\beta_0, \ldots, \beta_{2n-1}$ by setting $\varphi_1(\beta_i) = p_i$. Note that $\varphi_0(\alpha_0)$ may have several different expressions as a product of commutators, so we may choose any; even if some of the elements $p_1, \ldots, p_{2n-1}$ coincide we let all elements β_i to be distinct. Now we repeat the above construction for every homomorphism $\varphi_1|_{\langle\beta_i\rangle}$ of the free group on one generator to P and thus obtain a homomorphism $\varphi_2 : F_2 \to P$. Repeating the above construction we obtain a direct system of inclusions of free groups $F_1 \to F_2 \to F_3 \to \cdots$ and homomorphisms $\varphi_n : F_n \to P$. The direct limit of F_n is a locally free perfect group D and every group obtained by the above construction is called a grope group (and its clasifying space is a grope). This construction shows therefore that every homomorphism from a free group on one

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generator to a perfect group P can be extended to a homomorphism from a grope group to P. Note that in case the perfect group P has the Ore property ([8], [6]) that every element in P is a commutator, in the above process $(*)$ we can make every generator in the chosen basis of F_n a single commutator of two basis elements of F_{n+1}. The group obtained in this way is the minimal grope group M. Clearly every grope group admits many epimorphisms onto M. In the sequel we show that M admits a nontrivial homomorphism to another grope group G only if the latter is the free product $G \cong M * K$ where K is a grope group.

Gropes were introduced by Štan'ko [9]. They have an important role in geometric topology ([3], for more recent use in dimension theory see [5] and [4]). Their fundamental groups were used by Berrick and Casacuberta to show that the plus-construction in algebraic K-theory is localization [2]. Recently [1] such a group has appeared in the construction of a perfect group with a nonperfect localization.

In the first part of the paper we give a systematic definition of grope groups and prove some technical lemmas. In the second part we prove that the minimal grope group admits nontrivial homomorphisms to almost no other grope group thus proving that there exist at least two distinct grope groups.

2. Systematic definition of grope groups and basic facts

For every positive integer n let $\underline{n} = \{0, 1, \ldots, n-1\}$. The set of non-negative integers is denoted by $\mathbb{N}$. We denote the set of finite sequences of elements of a set X by $Seq(X)$ and the length of a sequence $s \in Seq(X)$ by $lh(s)$. The empty sequence is denoted by $\emptyset$. For $s, t \in Seq(X)$ let the concatenation be $s\,t \in Seq(X)$.

For a non-empty set A let $L(A)$ be the set $\{a, a^- : a \in A\}$, which we call the set of letters. We identify $(a^-)^-$ with a. Let $\mathcal{W}(A) = Seq(L(A))$, which we call the set of words. For a word $W \equiv a_0 \cdots a_n$, define $W^- \equiv a_n^- \cdots a_0^-$. We write $W \equiv W'$ for identity in $\mathcal{W}(A)$ while $W = W'$ for identity in the free group generated by A. For instance $a\,a^- = \emptyset$ but $aa^- \not\equiv \emptyset$. We adopt $[a, b] = a\,b\,a^{-1}\,b^{-1}$ as the definition of a commutator.

To describe all the grope groups we introduce some notation.

A *grope frame* S is a subset of $Seq(\mathbb{N})$ satisfying: $\emptyset \in S$ and for every $s \in S$ there exists $n > 0$ such that $\underline{2n} = \{i \in \mathbb{N} : si \in S\}$ and we wite $\omega(s) = 2n - 1$. If there is no ambiguity we write $\omega = \omega(s)$.

For each grope frame S we induce formal symbols c_s^S for $s \in S$ and define $E_m^S = \{c_s^S : lh(s) = m, s \in S\}$ and a free group $F_m^S = \langle E_m^S \rangle$.

Then define $e^S_m : F^S_m \to F^S_{m+1}$ by: $e^S_m(c^S_s) = [c^S_{s0}, c^S_{s1}] \cdots [c^S_{s\,\omega-1}, c^S_{s\,\omega}]$. Let $G^S = \varinjlim(F^S_m, e^S_m : m \in \mathbb{N})$ and every such group G^S is a grope group.

For $s \in S$, s is *binary branched*, if $\{i \in \mathbb{N} : si \in S\} = \underline{2}$. Let S_0 be a grope frame such that every $s \in S_0$ is binary branched, i.e. $S_0 = Seq(\underline{2})$. Then $G^{S_0} = M$ is the so-called minimal grope group. Since e^S_m is injective, we frequently regard F^S_m as a subgroup of G^S.

For a non-empty word W let head(W) of W be the left most letter b of W, i.e. $W \equiv bX$ for some word X, and let tail(W) of W be the right most letter c of W, i.e. $W \equiv Yc$ for some word Y.

For short we write the composite $e^S_{m,n} = e^S_{n-1} \cdots e^S_m$ for $m \leq n$. For a word $W \in \mathcal{W}(E^S_m)$ and $n \geq m$, the word $e^S_{m,n}[W] \in \mathcal{W}(E^S_n)$ can be expressed inductively as follows: $e^S_{m,m}[W] \equiv W$ and $e^S_{m,n+1}[W]$ is obtained by replacing every c_t in $e^S_{m,n}[W]$ by

$$\text{(P0)} \quad c^S_{t0}c^S_{t1}c^{S-}_{t0}c^{S-}_{t1} \cdots c^S_{t\,\omega-1}c^S_{t\,\omega}c^{S-}_{t\,\omega-1}c^{S-}_{t\,\omega}$$

and every c^{S-}_t by

$$\text{(P1)} \quad c^S_{t\,\omega}c^S_{t\,\omega-1}c^{S-}_{t\,\omega}c^{S-}_{t\,\omega-1} \cdots c^S_{t1}c^S_{t0}c^{S-}_{t1}c^{S-}_{t0}$$

respectively.

We drop the superscript ^{S}, if no confusion can occur.

For a reduced word $W \in \mathcal{W}(E_n)$ with $W \in F_m$ for $m < n$, let $W_0 \in \mathcal{W}(E_m)$ such that $e_{m,n}[W_0] \equiv W$. (The existence of W_0 is assured in Lemma 2.4.) A subword V of W is *small*, if there exists a letter c_s or c^-_s in W_0 and $i \in \mathbb{N}$ such that V is a subword of $e_{m+1,n}[c_{si}]$ or $e_{m+1,n}[c^-_{si}]$ respectively. Note that being small depends on m: in the identities in the minimal grope group

$$c_\emptyset = c_0c_1c^-_0c^-_1 = c_{00}c_{01}c^-_{00}c^-_{01}c_{10}c_{11}c^-_{10}c^-_{11}c_{01}c_{00}c^-_{01}c^-_{00}c_{11}c_{10}c^-_{11}c^-_{10} \equiv W$$

we see that $V \equiv c_{00}c_{01}$ is a small subword of W for $m = 0$, i.e. $W_0 = c_\emptyset$, and it is not a small subword of W for $m = 1$, i.e. $W_0 = c_0c_1c^-_0c^-_1$. In case $n = m + 1$ the small subwords of W are exactly all letters in W. In the following usage of the expression small subword the numbers m and n are always fixed in advance.

Observation 2.1. *Let $n > m + 1$ and let $W \equiv e_{m+1,n}[c_{s0}]$. Suppose that $X \in \mathcal{W}(E_n)$ is a reduced word and $X \in F_m$. When W is a subword of X, W may appear in*

$$\text{(C0)} \quad e_{m,n}[c_s] = e_{m+1,n}[c_{s0}c_{s1}c^-_{s0}c^-_{s1} \cdots c_{s\omega-1}c_{s\omega}c^-_{s\omega-1}c^-_{s\omega}]$$

or

$$\text{(C1)} \quad e_{m,n}[c^-_s] = e_{m+1,n}[c_{s\omega}c_{s\omega-1}c^-_{s\omega}c^-_{s\omega-1} \cdots c_{s1}c_{s0}c^-_{s1}c^-_{s0}].$$

The successive letter to W in (C0) is $\mathrm{head}(e_{m+1,n}[c_{s1}]) = c_{s10\ldots0}$*, but in (C1) it is* $\mathrm{head}(e_{m+1,n}[c_{s1}^-]) = \mathrm{head}(e_{m+2,n}[c_{s1\omega(s1)}]) = c_{s1\omega(s1)0\ldots0}$*. Thus the successive letter to W in X is not uniquely determined. However, if $X \equiv WY$ for some Y, the case (C1) can not appear, so the head of Y is uniquely determined as $c_{s10\cdots0}$.*

Similarly, the preceding letter to W is not uniquely determined – there are four possibilities:

- $\mathrm{tail}(e_{m+1,n}[c_{s1}]) = \mathrm{tail}(e_{m+2,n}[c_{s1\omega(s1)}^-]) = \mathrm{tail}(e_{m+3,n}[c_{s1\omega(s1)0}^-]) = c_{s1\omega(s1)0\ldots0}^-$ *in case (C1)*
- *If $X \equiv WY$ for some Y, there is no preceding letter to W*
- $\mathrm{tail}(e_{m,n}[c_t]) = \mathrm{tail}(e_{m+1,n}[c_{t\omega}^-]) = \mathrm{tail}(e_{m+2,n}[c_{t\omega0}^-]) = c_{t\omega0\ldots0}^-$ *in case (C1) if $X = e_{m+1,n}[Zc_tc_sY]$ for some Z, Y*
- $\mathrm{tail}(e_{m,n}[c_t^-]) = \mathrm{tail}(e_{m+1,n}[c_{t0}^-]) = c_{t0\ldots0}^-$ *in case (C1) if $X = e_{m+1,n}[Zc_t^-c_sY]$ for some Z, Y.*

However, the preceding letter to W determines the succesive letter to W uniquely:

- *If the preceding letter to W is $c_{s1\omega(s1)0\ldots0}^-$ then we are in (C1)*
- *In all other cases we are in (C0).*

Observation 2.2. *A letter $c_{s0\cdots0} \in \mathcal{W}(E_n)$ for $lh(s) = m$ possibly appears in $e_{m,n}[W_0]$ in the following cases. When $n = m+1$, c_{s0} appears once in $e_{m,n}[c_s]$ and also once in $e_{m,n}[c_s^-]$. According to the increase of n, $c_{s0\cdots0}$ appears in many parts. $c_{s0\cdots0}$ appears 2^{n-m-1}-times in $e_{m,n}[c_s]$ and also 2^{n-m-1}-times in $e_{m,n}[c_s^-]$.*

Lemma 2.3. *For a word $W \in \mathcal{W}(E_m)$ and $n \geq m$, $e_{m,n}[W]$ is reduced, if and only if W is reduced.*

Lemma 2.4. *For a reduced word $V \in \mathcal{W}(E_n)$ and $n \geq m$, $V \in F_m$ if and only if there exists $W \in \mathcal{W}(E_m)$ such that $e_{m,n}[W] \equiv V$.*

Proof. The sufficiency is obvious. To see the other direction, let W be a reduced word in $\mathcal{W}(E_m)$ such that $e_{m,n}[W] = V$ in F_n. By Lemma 2.3 $e_{m,n}[W]$ is reduced. Since every element in F_n has a unique reduced word in $\mathcal{W}(E_n)$ presenting itself, we have $e_{m,n}[W] \equiv V$. □

Lemma 2.5. *Let $m < n$ and A be a non-empty word in $\mathcal{W}(E_n)$. Let X_0AY_0 and X_1AY_1 be reduced words in $\mathcal{W}(E_n)$ satisfying $X_0AY_0, X_1AY_1 \in F_m$.*

1. *If A is not small, $X_0A \notin F_m$ and $X_1A \notin F_m$, then the heads of Y_0 and Y_1 are the same.*
2. *Let X_0 be an empty word. If A is not small and $A \notin F_m$, the heads of Y_0 and Y_1 are the same.*

(3) *Let X_0 and X_1 be empty words. If $A \notin F_m$, the heads of Y_0 and Y_1 are the same.*

Proof. (1) Since $X_0AY_0 \in F_m$ but $X_0A \notin F_m$, we have a letter $c \in E_m \cup E_m^-$ and words U_0, U_1, U_2 such that $U_1 \not\equiv \emptyset$, $U_2 \not\equiv \emptyset$, $X_0A \equiv U_0U_1$ and $U_1U_2 \equiv e_{m,n}[c]$. Since A is not small, c and U_0, U_1, U_2 are uniquely determined by A. Since the same thing holds for X_1AY_1, we have the conclusion by Observation 1 for $n > m+1$. (The case for $n = m+1$ is easier.)

(2) Since $AY_0 \in F_m$, $A \notin F_m$ and A is not a small word, for any word B such that BA is reduced we have $BA \notin F_m$. In particular $X_1A \notin F_m$ and the conclusion follows from (1).

(3) Since $AY_0 \in F_m$, there are A_0 and non-empty U_0, U_1 such that $A_0 \in F_m$, $A \equiv A_0U_0$ and $U_0U_1 \equiv e_{m,n}[c]$ for some $c \in E_m \cup E_m^-$. Since $A \notin F_m$, the head of U_1 is uniquely determined by A and hence the heads of Y_0 and Y_1 are the same (Observation 1). $\square$

Lemma 2.6. *Let $m < n$ and A, X, Y in $\mathcal{W}(E_n)$ and $AXA^-Y \in F_m$. If AXA^-Y is reduced and A is not small, then $AXA^- \in F_m$ and $Y \in F_m$.*

Proof. The head of the reduced word in $\mathcal{W}(E_m)$ for the element AXA^-Y is c_s or c_s^- for $c_s \in E_m$. According to c_s or c_s^-, $A \equiv e_{m+1,n}[c_{s0}]Z$ or $e_{m+1,n}[c_{sk}]Z$ for a non-empty word Z, where $\underline{k+1} = \{i \in \mathbb{N} : si \in S\}$ is even. Then $A^- \equiv Z^-e_{m+1,n}[c_{s0}^-]$ or $A^- \equiv Z^-e_{m+1,n}[c_{sk}^-]$ and hence $AXA^- \in F_m$ and consequently $Y \in F_m$. $\square$

Lemma 2.7. *For $e \neq x \in F_m^S$ and $u \in G^S$, $uxu^{-1} \in F_m^S$ implies $u \in F_m^S$.*

Proof. There exists $n \geq m$ such that $u \in F_n$. Let W be a cyclically reduced word and V be a reduced word such that $x = VWV^-$ in F_m and VWV^- is reduced. Then $e_{m,n}(x) = e_{m,n}[V]e_{m,n}[W]e_{m,n}[V]^-$ and $e_{m,n}[V]$ is reduced and $e_{m,n}[W]$ is cyclically reduced by Lemma 2.3. Let U be a reduced word for u in F_n. Let $k = lh(U)$. Then $e_{m,n}(x^{2k+1}) = e_{m,n}[V]e_{m,n}[W]^{2k+1}e_{m,n}[V]^-$ and the right hand term is a reduced word. Hence the reduced word for ux^ku^- of the form $Xe_{m,n}[W]Y$, where $Ue_{m,n}[V]e_{m,n}[W]^k = X$ and $e_{m,n}[W]^ke_{m,n}[V]^-U^- = Y$. Since $ux^ku^{-1} \in F_m$, $X \in F_m$ and $Y \in F_m$. Now we have $Ue_{m,n}[V] \in e_{m,n}(F_m)$ and hence $U \in e_{m,n}(F_m)$, which implies the conclusion. $\square$

Lemma 2.8. *Let UWU^- be a reduced word in $\mathcal{W}(E_n)$. If $UWU^- \in F_m$ and W is cyclically reduced, then $U, W \in F_m$.*

Proof. If U is empty or $n = m$, then the conclusion is obvious. If $U \in F_m$, then $WU^- \in F_m$ and so $W \in F_m$. Suppose that U is

$U \notin F_m$. Since $UWU^-, UW^-U^- \in F_m$, the head of W and that of W^- is the same by Lemma 2.5 (3), which contradicts that W is cyclically reduced. □

Lemma 2.9. *Let XY and YX be reduced words in $\mathcal{W}(E_n)$ for $n \geq m$. If XY and YX belong to F_m, then both of X and Y belong to F_m.*

Proof. We may assume $n > m$. When $n > m$, the head of $e_{m,n}[W]$ for a non-empty word $W \in \mathcal{W}(E_m)$ is $c_{s0\cdots 0}$ or $c_{sk0\cdots 0}$ where $lh(s) = m$ and $k+1 = \{i \in \mathbb{N} : si \in S\}$ is even. (When $n = m+1$, there appears no $0\cdots 0$.) Since $X^-Y^- \in F_m$ and X^-Y^- is reduced, the tail of X is of the form $c^-_{s0\cdots 0}$ or $c^-_{sk0\cdots 0}$. We only deal with the former case. Suppose that $X \notin F_m$. Since $XY \in F_m$ and XY is reduced, $X \equiv Ze_{m+1,n}[c_{s1}c^-_{s0}]$ for some Z. This implies $X^- \equiv e_{m+1,n}[c_{s0}c^-_{s1}]Z^-$, which contradicts that $X^-Y^- \in F_m$ and X^-Y^- is reduced. Now we have $X, Y \in F_m$. □

Lemma 2.10. *Let $m < n$ and A, B, C in $\mathcal{W}(E_n)$ and $e \neq ABCA^-B^-C^- \in F_m$. If $ABCA^-B^-C^-$ is a reduced word and at least one of A, B, C is not small, then $A, B, C \in F_m$.*

Proof. Since $ABCA^-B^-C^- \neq e$, at most one of A, B, C is empty. When C is empty, the conclusion follows from Lemma 2.6 and the fact that BAB^-A^- is also reduced and $BAB^-A^- \in F_m$.

Now we assume that A, B, C are non-empty. If A is not small, then $ABCA^- \in F_m$ and $B^-C^- \in F_m$ by Lemma 2.6. Since BC is cyclically reduced, $A \in F_m$ and $BC \in F_m$ by Lemma 2.8. The conclusion follows from Lemma 2.9. In the case that C is not small, the argument is similar. The remaining case is when A and C are small. Then $ABCA^-B^-C^- \in F_m$ and $CBAC^-B^-A^- \in F_m$ imply $A \equiv C$, which contradicts the reducedness of $ABCA^-B^-C^-$. □

Lemma 2.11. *Let $m < n$ and A, B, C in $\mathcal{W}(E_n)$ and $e \neq ABCA^-B^-C^- \in F_m$. If $ABCA^-B^-C^-$ is a reduced word and A, B, C are small, then one of A, B, C is empty.*

Assume C is empty. Then there exists $c_s \in E_m$ such that s is binary branched and either

$$A \equiv e_{m+1,n}[c_{s0}] \text{ and } B \equiv e_{m+1,n}[c_{s1}],$$

or

$$A \equiv e_{m+1,n}[c_{s1}] \text{ and } B \equiv e_{m+1,n}[c_{s0}].$$

Proof. Since A, B, C are small, all the words A, B, C and their inverses must be subwords of $e_{m+1,n}[c_{si}]$, $i = 0, 1$, or $e_{m+1,n}[c^-_{si}]$, for an element $c_s \in E_m$, and in particular that either

$$ABCA^-B^-C^- = e_{m,n}(c_s) = e_{m+1,n}[c_{s0}c_{s1}c^-_{s0}c^-_{s1}]$$

or

$$ABCA^-B^-C^- = e_{m,n}(c_s^-) = e_{m+1,n}[c_{s1}c_{s0}c_{s1}^-c_{s0}^-],$$

where the left most and right most terms are reduced words. We remark that if the cardinality of $\{i \in \mathbb{N} : si \in S\}$ were greater than 2, one of A, B, C would not be small; hence in our case s is binary branched.

We only deal with the first case. Then $ABC \equiv e_{m+1,n}[c_{s0}c_{s1}]$ and $A^-B^-C^- \equiv e_{m+1,n}[c_{s0}^-c_{s1}^-]$. In case A, B, C are non-empty, A is a proper subword of $e_{m+1,n}[c_{s0}]$ or C is a proper subword of $e_{m+1,n}[c_{s1}]$. In either case $A^-B^-C^- \equiv e_{m+1,n}[c_{s0}^-c_{s1}^-]$ does not hold. Hence one of A, B, C is empty. We may assume C is empty. Since A, B are small, $A \equiv e_{m+1,n}[c_{s0}]$ and $B \equiv e_{m+1,n}[c_{s1}]$. □

3. Proof of Theorem 3.1

In this section we prove

Theorem 3.1. *The minimal grope group $M = G^{S_0}$ admits a nontrivial homomorphism into a grope group G^S, if and only if there exists $s \in S$ such that a frame $\{t \in Seq(\mathbb{N}) : st \in S\}$ is equal to S_0.*

It is easy to see that the condition on G^S in the above theorem is equivalent to $G^S \cong M * K$, where K is another grope group.

In our proof of Lemma 3.9 we analyze a reduction procedure of a word $Y^-ABYX^-A^-B^-X$ where Y^-ABY and $X^-A^-B^-X$ are reduced. Lemmas 3.2, 3.3, 3.4 and 3.5 show connections between our reduction steps in case at least one of X and Y is empty. Lemma 3.6 corresponds to the final step, i.e. when we have the reduced word. Lemmas 3.7 and 3.8 correspond to the case that X and Y are non-empty. In the following lemmas we assume $m < n$.

Lemma 3.2. *Let $A, B \in \mathcal{W}(E_n)$ be non-empty reduced words such that $ABA^-B^- \neq e$ and AB and A^-B^- are reduced words. Then the following hold:*

(1.1) *If $B \equiv B_0A$, then B_0 is non-empty, AB_0 and $A^-B_0^-$ are reduced words and $AB_0A^-B_0^- = ABA^-B^-$. In addition if $AB_0, A^-B_0^- \in F_m$, then $AB, A^-B^- \in F_m$.*

(1.2) *If $A \equiv A_0B$, then A_0 is non-empty, A_0B and $A_0^-B^-$ are reduced words and $A_0BA_0^-B^- = ABA^-B^-$. In addition if $A_0B, A_0^-B^- \in F_m$, then $AB, A^-B^- \in F_m$.*

(1.3) *If $A \equiv A_0Z$ and $B \equiv B_0Z$ for non-empty words A_0 and B_0 and $B_0A_0^-$ is reduced, then $A_0ZB_0A_0^-Z^-B_0^-$ is reduced and $A_0ZB_0A_0^-Z^-B_0^- = ABA^-B^-$. In addition if $A_0, B_0, Z \in F_m$, then $AB, A^-B^- \in F_m$.*

Proof. We only show (1.1). The non-emptiness of B_0 follows from $ABA^-B^- \neq e$. Since AB and A^-B^- are reduced, AB_0 and $A^-B_0^-$ are cyclically reduced and hence the second statement follows from Lemma 2.9. □

Lemma 3.3. *Let $A, B, C \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $ABCA^-B^-C^- \neq e$ and AB and $CA^-B^-C^-$ are reduced words. Then the following hold:*

(2.1) *If $B \equiv B_0C^-$, then AB_0 and $A^-CB_0^-C^-$ are reduced words and $AB_0A^-CB_0^-C^- = ABCA^-B^-C^-$. In addition if $AB_0A^-, CB_0^-C^- \in F_m$, then $AB, CA^-B^-C^- \in F_m$.*

(2.2) *If $C \equiv B^-C_0$, then AC_0 and $A^-B^-C_0^-B$ are reduced words and $AC_0A^-B^-C_0^-B = ABCA^-B^-C^-$. In addition if $AC_0A^-, B^-C_0^-B \in F_m$, then $AB, CA^-B^-C^- \in F_m$.*

(2.3) *If $B \equiv B_0Z^-$ and $C \equiv ZC_0$ for non-empty words B_0 and C_0 and B_0C_0 is reduced, then $AB_0C_0A^-ZB_0^-C_0^-Z^-$ is reduced and $AB_0C_0A^-ZB_0^-C_0^-Z^- = ABCA^-B^-C^-$. In addition if $AB_0C_0A^-, ZB_0^-C_0^-Z^- \in F_m$, then $AB, CA^-B^-C^- \in F_m$.*

Proof. (2.1) The first proposition is obvious. Let $B_0 \equiv XB_1X^-$ for a cyclically reduced word B_1. Since $(AX)B_1(AX)^-, (CX)B_1^-(CX)^- \in F_m$, $AX, CX, B_1 \in F_m$ by Lemma 2.8. Now $AB = (AX)B_1(CX)^- \in F_m$ and $CA^-B^-C^- = (CX)(AX)^-(CB_0^-C^-) \in F_m$. We see (2.2) similarly.

For (2.3) observe the following. Since the both B_0 and C_0 are non-empty, B_0C_0 and $B_0^-C_0^-$ are cyclically reduced. Hence, using Lemmas 2.8 and 2.9, we have (2.3). □

The next two lemmas are straightforward and we omit the proofs.

Lemma 3.4. *Let $A, B, C \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $ABA^-CB^-C^- \neq e$ and AB and $A^-CB^-C^-$ are reduced. Then the following hold:*

(3.1) *If $A \equiv A_0B$, then A_0B and $A_0^-CB^-C^-$ are reduced and $A_0BA_0^-CB^-C^- = ABA^-CB^-C^-$. In addition if $A_0BA_0^-, CB^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(3.2) *If $B \equiv B_0A$, then AB_0 and $CA^-B_0^-C^-$ are reduced and $AB_0CA^-B_0^-C^- = ABA^-CB^-C^-$. In addition if $AB_0, CA^-B_0^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(3.3) *If $B \equiv B_0Z$ and $A \equiv A_0Z$ for non-empty words A_0 and B_0 and $B_0A_0^-$ is reduced, then $A_0ZB_0A_0^-CZ^-B_0^-C^-$ is reduced. In addition if $A_0ZB_0A_0^-, CZ^-B_0^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

Lemma 3.5. *Let $A, B, C \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $ABA^-CB^-C^- \neq e$ and A and $BA^-CB^-C^-$ are reduced words. Then the following hold:*

(4.1) *If $A \equiv A_0B^-$, A_0 and $BA_0^-CB^-C^-$ are reduced and $A_0BA_0^-CB^-C^- = ABA^-CB^-C^-$. In addition if $A_0BA_0^-, CB^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(4.2) *If $B \equiv A^-B_0$, and $B_0A^-CB_0^-AC^-$ is reduced and $B_0A^-CB_0AC^- = ABA^-CB^-C^-$. In addition if $B_0A^-, CB_0^-AC^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(4.3) *If $A \equiv A_0Z^-$ and $B \equiv ZB_0$ for non-empty words A_0, B_0 and A_0B_0 is reduced, then $A_0B_0ZA_0^-CB_0^-Z^-C^-$ is reduced and $A_0B_0ZA_0^-CB_0^-Z^-C^- = ABA^-CB^-C^-$. In addition if $A_0B_0ZA_0^-$, $CB_0^-Z^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

Lemma 3.6. *Let $A, B, C, D \in \mathcal{W}(E_n)$ be reduced non-empty words.*

(1) *if ABA^-B^- is reduced and $ABA^-B^- \in F_m$ and at least one of A, B is not small, then $A, B \in F_m$;*
(2) *if $ABCA^-B^-C^-$ is reduced and $ABCA^-B^-C^- \in F_m$ at least one of A, B, C is not small, then $A, B, C \in F_m$;*
(3) *if $CABC^-DA^-B^-D^-$ is reduced and $CABC^-DA^-B^-D^- \in F_m$, then $A, B, C, D \in F_m$.*
(4) *if $CAC^-DA^-D^-$ is reduced and $CAC^-DA^-D^- \in F_m$, then $CAC^-, DA^-D^- \in F_m$.*

Proof. The statements (1) and (2) are paraphrases of Lemma 2.10.
(3) Let c be the head of C and d be the tail of D^-. Since c^- and d^- are contiguous, we have $CABC^-, DA^-B^-D^- \in F_m$. Since AB and A^-B^- are reduced and the both A and B are non-empty, AB is cyclically reduced. Now the conclusion follows from Lemmas 2.8 and 2.9.
(4) This follows from a reasoning in the proof of (3). □

Lemma 3.7. *Let A^-B^- and $X_0ABX_0^-$ be reduced words such that $X_0AB \equiv BAX_1$ for some X_1. If $lh(X_0) \leq lh(B)$, then there exist A', B' such that $lh(B') < lh(B)$, $(A')^-(B')^-$ and $X_0A'B'X_0^-$ are reduced words, $X_0A'B' \equiv B'A'X_1$, $A^-B^-X_0ABX_0^- = (A')^-(B')^-X_0A'B'X_0^-$, and $A, B \in \langle X_0, A', B' \rangle$.*

Proof. First we remark that $lh(X_0) \neq lh(B)$ since BX_0^- is reduced. Hence $lh(B) > lh(X_0)$. If $lh(B) = lh(X_0) + lh(A)$, then we have $X_0A \equiv B \equiv AX_1$ and have the conclusion, i,e, $A' \equiv A$ and $B' \equiv \emptyset$.

If $lh(B) < lh(X_0) + lh(A)$, we have $k > 0$ and A_0, A_1 such that $B \equiv X_0A_0A_1$, $A \equiv (A_0A_1)^kA_0$, and A_1 is non-empty. (We remark that A_0 may be empty.) Let $A' \equiv A_0$ and $B' \equiv A_1$. Since $lh(X_0) + lh(A) =$

$lh(B)+(k-1)lh(A_0A_1)+lh(A_0)$, we have $B\equiv A_1A_0X_1$. Let $A'\equiv A_0$ and $B'\equiv A_1$, then we have the conclusion.

If $lh(B)>lh(X_0)+lh(A)$, we have $k>0$ and B_0,B_1 such that $B_0B_1\equiv X_0A$, $B\equiv(B_0B_1)^kB_0$, and B_1 is non-empty. (We remark that B_0 may be empty.) Since $lh(B_1B_0)=lh(AX_1)$, we have $B_1B_0\equiv AX_1$. Now $B\equiv X_0A(B_0B_1)^{k-1}B_0\equiv(B_0B_1)^{k-1}B_0AX_1$ holds. Let $A'\equiv A$ and $B'\equiv(B_0B_1)^{k-1}$, then we have the conclusion. □

In Lemma 3.7 we have $A^-B^-X_0ABX_0^-=X_1X_0^-=(A')^-(B')^-X_0A'B'X_0^-$.

Lemma 3.8. *Let $A,B,X,Y\in\mathcal{W}(E_n)$ be reduced words (possibly empty) such that X and Y are non-empty, $Y^-A^-B^-YX^-ABX\neq e$, $Y^-A^-B^-Y$ and X^-ABX are reduced words, and the reduced word of $Y^-A^-B^-YX^-ABX$ is cyclically reduced.*

If $Y^-A^-B^-YX^-ABX\in F_m$, then

(1) *$Y^-A^-B^-Y,X^-ABX\in F_m$, or*
(2) *$Y^-A^-B^-YX^-ABX$ is equal to c_s or c_s^- for some s such that $lh(s)=m$ and s is binary branched.*

Proof. If YX^- is reduced, then $Y^-A^-B^-YX^-ABX$ is cyclically reduced. By an argument analyzing the head and the tail of Y^- and X we can see $Y^-A^-B^-Y,X^-ABX\in F_m$.

Otherwise, in the cancellation of $Y^-A^-B^-YX^-ABX$ the leftmost Y^- or the rightmost X is deleted. Since $Y^-A^-B^-YX^-ABX\neq e$ and $lh(Y^-A^-B^-Y)=2lh(Y)+lh(AB)$ and $lh(X^-ABX)=2lh(X)+lh(AB)$, $lh(X)\neq lh(Y)$. We suppose that $lh(X)>lh(Y)$, i.e. the head of Y^- is deleted. Then we have $X\equiv ZY$ for a non-empty word Z.

We first analyze a reduced word of $A^-B^-Z^-ABZ$, where A^-B^- is deleted. The head part of Z^-AB is BA. Applying Lemma 3.7 for $X_0\equiv Z^-$ and X_1 repeatedly, we have reduced words A_0 and B_0 such that $Z^-A_0B_0Z$ is reduced, $Z^-A_0B_0\equiv B_0A_0X_1$ for some X_1, $A_0^-B_0^-Z^-A_0B_0Z=A^-B^-Z^-ABZ$, $A,B\in\langle Z,A_0,B_0\rangle$ and $lh(B_0)<lh(Z)$.

It never occurs that the both A_0 and B_0 are empty, but one of A_0 and B_0 may be empty. If $B_0=\emptyset$, interchange the role of A_0 and B_0 and by Lemma 3.7 we can assume B_0 is non-empty and $lh(B_0)<lh(Z)$.

First we deal with the case A_0 is empty. Since the left most B_0^- is deleted in the reduction of $B_0^-Z^-B_0Z$, we have non-empty Z_0 such that $Z\equiv Z_0B_0^-$ and have a reduced word $Z_0^-B_0Z_0B_0^-$ with $Z_0^-B_0Z_0B_0^-=B_0^-Z^-B_0Z$. Since the left most Y^- is deleted in the reduction of $Y^-B_0^-Z^-B_0ZY$ and $Z_0^-B_0Z_0B_0^-Y$ is reduced, $Z_0^-B_0Z_0B_0^-$ is cyclically reduced and hence the reduced word of $Y^-A^-B^-YX^-ABX$ is a cyclical transformation of $Z_0^-B_0Z_0B_0^-$. By the fact that Y is the head part of $B_0^-Z^-B_0ZY$, Y is of the form $(Z_0^-B_0Z_0B_0^-)^kY_0$ where $Y_0Y_1 \equiv Z_0^-B_0Z_0B_0^-$ for some non-empty Y_1 and $k \geq 0$.

If Y_0 is empty, we have $Y^-A^-B^-YX^-ABX = Z_0^-B_0Z_0B_0^-$. If one of Z_0 and B_0 is not small, then $Z_0, B_0 \in F_m$ by Lemma 2.10 and we have $Y^-A^-B^-Y, X^-ABX \in F_m$ by Lemma 3.7 and the fact $Y = (Z_0^-B_0Z_0B_0^-)^k$. Otherwise, i.e., when of Z_0 and B_0 are small, $Y^-A^-B^-YX^-ABX = Z_0^-B_0Z_0B_0^-$ is equal to c_s or c_s^- for some s such that $lh(s) = m$ and s is binary branched by Lemma 2.11.

If $Y_0 \equiv Z_0^-$, $Y_0 \equiv Z_0^-B_0$ or $Y_0 \equiv Z_0^-B_0Z_0$, the argument is similar to the case that Y_0 is empty. Otherwise Y_0 cut short Z_0^-, B_0, Z_0 or B_0^-. Since arguments are similar, we only deal with the case that $Y_0 \equiv Z_0^-B_1$ where $B_1B_2 \equiv B_0$ for non-empty B_1 and B_2. Then $Y^-A^-B^-YX^-ABX = B_2Z_0B_2^-B_1^-Z_0^-B_1$ and hence $B_2Z_0B_2^-, B_1^-Z_0^-B_1 \in F_m$ by Lemma 3.6 (4). Let Z_1 be a cyclically reduced word such that $Z_0 \equiv U^-Z_1U$. Then $Z_1, B_2U^-, UB_1 \in F_m$ by Lemma 2.8. Now

$$\begin{aligned}
Y^-Z_0Y &= B_1^-Z_0(B_1B_2Z_0^-B_2^-B_1^-Z_0)^kZ_0(Z_0^-B_1B_2Z_0B_2^-B_1^-)^kZ_0^-B_1 \\
&= (B_1^-Z_0B_1B_2Z_0^-B_2^-)^kB_1^-Z_0B_1(B_2Z_0B_2^-B_1^-Z_0^-B_1)^k \\
Y^-B_0Y &= B_1^-Z_0(B_1B_2Z_0^-B_2^-B_1^-Z_0)^kB_1B_2(Z_0^-B_1B_2Z_0B_2^-B_1^-)^kZ_0^-B_1 \\
&= B_1^-Z_0B_1(B_2Z_0^-B_2^-B_1^-Z_0B_1)^kB_2Z_0^-B_1(B_2Z_0B_2^-B_1^-Z_0^-B_1)^k.
\end{aligned}$$

Hence $Y^-Z_0Y, Y^-B_0Y \in F_m$. Since $Z = Z_0B_0^-$ and $A, B \in \langle Z, B_0\rangle$, we have $Y^-ABY, X^-A^-B^-X \in F_m$.

Next we suppose that A_0 is non-empty. We have $k > 0$ and A_1 and A_2 such that $Z^- \equiv B_0A_1A_2$, $A_0 \equiv (A_1A_2)^kA_1$, $X_1 \equiv A_2A_1B_0$. Since $X^-AB \equiv UX_1$ for some U and X^-ABZ is reduced, $X_1Z \equiv A_2A_1B_0A_2^-A_1^-B_0^-$ is a reduced word. By the assumption a reduced word of $Y^-A_2A_1B_0A_2^-A_1^-B_0^-Y$ is cyclically reduced and $A_2A_1B_0A_2^-A_1^-B_0^-Y$ is reduced, hence $X_1Z \equiv A_2A_1B_0A_2^-A_1^-B_0^-$ is cyclically reduced and the reduced word of $Y^-A_2A_1B_0A_2^-A_1^-B_0^-Y$ is given by a cyclical transformation of $A_2A_1B_0A_2^-A_1^-B_0^-$. Hence $Y \equiv (A_2A_1B_0A_2^-A_1^-B_0^-)^kY_0$ where $k \geq 0$ and $A_2A_1B_0A_2^-A_1^-B_0^- \equiv Y_0Y_1$ for some Y_1.

For instance the reduced word of $Y^-A_2A_1B_0A_2^-A_1^-B_0^-Y$ is of the form $B_0A_2^-A_1^-B_0^-A_2A_1$ or $B_2A_2^-A_1^-B_2^-B_1^-A_2A_1B_1$ where $B_0 \equiv B_1B_2$. By Lemma 3.6 (4) or (3) respectively we conclude $A_1, A_2, B_0 \in F_m$ or $A_1, A_2, B_1, B_2 \in F_m$ which implies $Y^-ABY, X^-A^-B^-X \in F_m$. □

Lemma 3.9. *For every grope group G^S the following hold:*

If $e \neq [u, v] \in F_m$ and at least one of u and v does not belong to F_m, then $[u, v]$ is conjugate to c_s or c_s^- in F_m for some s such that $lh(s) = m$ and s is binary branched.

Proof. We have $n > m$ such that $u, v \in F_n$. It suffices to show the lemma in case that the reduced word for $[u, v]$ is cyclically reduced. For, suppose that we have the conclusion of the lemma in the indicated case. Let $[u, v] \in F_m$ and $[u, v] = XYX^-$ where XYX^- is a reduced word and Y is cyclically reduced. Then we have $[X^-uX, X^-vX] = X^-[u, v]X = Y$. On the other hand $X, Y \in F_m$ by Lemma 2.8. By the assumption at least one of X^-uX and X^-vX does not belong to F_m. Since $[u, v]$ is conjugate to Y in F_m, we have the conclusion.

Let $u, v \in F_n$ such that $[u, v] \neq e$ and the reduced word for $[u, v]$ is cyclically reduced. There exist a cyclically reduced non-empty word $V_0 \in \mathcal{W}(E_n)$ and a reduced word $X \in \mathcal{W}(E_n)$ such that $v = X^-V_0X$ and the word X^-V_0X is reduced. Let U_0 be a reduced word for uX^-. Since V_0 is a cyclically reduced word, at least one of U_0V_0 and $V_0U_0^-$ is reduced. When U_0V_0 is reduced, there exist $k \geq 0$ and reduced words Y, A, B such that Y^-ABY is reduced, $U_0 \equiv Y^-AV_0^k$ and $V_0 \equiv BA$. When $V_0U_0^-$ is reduced, there exist $k \geq 0$ and reduced words Y, A, B such that Y^-ABY is reduced, $U_0 \equiv Y^-A(V_0^-)^k$ and $V_0 \equiv BA$. In the both bases $uvu^{-1} = Y^-ABY$ and $v = X^-BAX$. We remark that AB and BA are cyclically reduced.

We analyze a reduction procedure of $Y^-ABYX^-A^-B^-X$ in the following.

(Case 0): X and Y are empty.

In this case the both A and B are non-empty and corresponds to Lemma 3.2. Using (1.1) and (1.2) alternately and (1.3) possibly as the last step we obtain a reduced word of ABA^-B^-. If the reduced word $XYZX^-Y^-Z^-$ satisfies that one of X, Y, Z is not small, by (1) and (2) of Lemma 3.6 and applying Lemma 3.2 repeatedly we can see $A, B \in F_m$. Otherwise, one of X, Y, Z is empty and $[u, v] = c_s$ or $[u, v] = c_s^-$ for some binary branched s with $lh(s) = m$ by Lemma 2.11.

(Case 1): Y is empty, but X is non-empty.

(Case 2): X is empty, but Y is non-empty.

In these cases arguments are symmetric, we only deal with (Case 1). There is possibility that one of A and B may be empty, though at least one of A and B is non-empty. We assume that A is non-empty. We trace Lemmas 3.3, 3.4, 3.5 to get a reduced word of $ABX^-A^-B^-X$. Then we apply one of (2), (3) and (4) of Lemma 3.6 to the reduced word and applying Lemma 3.2 repeatedly we get a reduced word. Then we have $A, B \in F_m$, which implies $u, v \in F_m$, or $[u, v] = c_s$ etc. as in (Case 0).

(Case 3): The both X and Y are non-empty.

Only in this case we use the assumption that the reduced word of $Y^-ABYX^-A^-B^-X$ is cyclically reduced. By Lemma 3.8 we have the conclusion. □

Lemma 3.10. *Let F be a free group generated by C and $c, d \in C$ be distinct elements. If $[c, d] = [u, v]$ for $u, v \in F$, then neither u nor v belongs to the commutator subgroup of F.*

Proof. Since c, d are generators, $[c, d] \notin [F, [F, F]]$ and the conclusion follows. □

Lemma 3.11. *Let F be a free group generated by B and $b_0, b_1 \in B$ be distinct. If $c, d \in \{b, b^- : b \in B\}$ and $[b_0, b_1] = [x^{-1}cx, y^{-1}dy]$ for $x, y \in F$, then $c, d \in \{b_0, b_0^-, b_1, b_1^-\}$ and moreover $c \in \{b_0, b_0^-\}$ iff $d \in \{b_1, b_1^-\}$ and $c \in \{b_1, b_1^-\}$ iff $d \in \{b_0, b_0^-\}$.*

Proof. Using a canonical projection to $\langle b_0, b_1\rangle$ we easily see that $c, d \in \{b_0, b_0^-, b_1, b_1^-\}$. To see the remaining part it suffices to show that if $c = b_0$, and $d = b_0$ or b_0^-, then $[b_0, b_1] \neq [x^{-1}cx, y^{-1}dy]$ for any x, y.

We show that $b_0b_1b_0^-b_1^-$ is not cyclically equivalent to the reduced word for $[x^{-1}cx, y^{-1}dy]$. For this purpose we may assume $x = e$. We only deal with $d = b_0$. We have a reduced word Y such that $y^{-1}b_0y = Y^-b_0Y$ and Y^-b_0Y is reduced. (Note that $y = Y$ may not hold.) The head of Y is not b_0 nor b_0^-, since Y^-b_0Y is reduced. When the tail of Y is b_0 or b_0^-, we choose $n \geq 0$ so that $Y \equiv Zb_0^n$ or $Y \equiv Z(b_0^-)^n$ respectively and n is maximal. Then Z is non-empty. Now $b_0Z^-b_0Zb_0^-Z^-b_0^-Z$ is a cyclically reduced word which is cyclically equivalent to $b_0Y^-b_0Yb_0^-Y^-b_0^-Y$. Since $b_0Z^-b_0Zb_0^-Z^-b_0^-Z$ is not cyclically equivalent to $b_0b_1b_0^-b_1^-$, we have the conclusion. □

Proof of Theorem 3.1. Let $h : G^{S_0} \to G^S$ be a nontrivial homomorphism. Then there exists $s_* \in S_0$ such that $h(c_{s_*})$ is nontrivial (clearly for every finite sequence s starting with s_* also $h(c_s)$ is nontrivial). We let $c_s = c_s^{S_0}$ and $d_t = c_t^S$ and $F_m = F_m^S$.

We have n such that $h(c_{s_*}) \in F_n$. Since F_n is free, $\mathrm{Im}(h)$ is not included in F_n and hence there exists $s_0 \in S_0$ starting with s_* and such that $h(c_{s_0}) \in F_n$, but $h(c_{s_00}) \notin F_n$ or $h(c_{s_01}) \notin F_n$. Then by Lemma 3.9 we have $d_{t_0} \in E_n$ such that $h(c_{s_0})$ is conjugate to d_{t_0} or $d_{t_0}^-$ and t_0 is binary branched.

Moreover, Lemma 2.7 implies that neither $h(c_{s_00})$ nor $h(c_{s_01})$ belongs to F_n. We show the following by induction on $k \in \mathbb{N}$:

(1) For $u \in Seq(\underline{2})$ with $lh(u) = k$
   - (a) $h(c_{s_0u})$ is conjugate to d_{t_0v} or $d_{t_0v}^-$ in F_{n+k} and t_0v is binary branched for some $v \in Seq(\underline{2})$ with $lh(v) = k$;

(b) Neither $h(c_{s_0u0})$ nor $h(c_{s_0u1})$ belongs to F_{n+k};

(2) For every $v \in Seq(\underline{2})$ with $lh(v) = k$ there exists $u \in Seq(\underline{2})$ such that $lh(u) = k$ and $h(c_{s_0u})$ is conjugate to d_{t_0v} or $d^-_{t_0v}$ in F_{n+k}.

We have shown that this holds when $k = 0$.

Suppose that (1) and (2) hold for k. Let $lh(u) = k$ and $h(c_{s_0u})$ is conjugate to d_{t_0v} or $d^-_{t_0v}$ etc. Then $[h(c_{s_0u0}), h(c_{s_0u1})]$ is conjugate to $[d_{t_0v0}, d_{t_0v1}]$ or $[d_{t_0v1}, d_{t_0v0}]$ in F_{n+k+1}. We claim $h(c_{s_0u0}) \in F_{n+k+1}$. To show this by contradiction, suppose that $h(c_{s_0u0}) \notin F_{n+k+1}$. Apply Lemma 3.9 to F_{n+k+1}, then we have $[h(c_{s_0u0}), h(c_{s_0u1})]$ is a conjugate to d_t or d^-_t with $lh(t) = n + k + 1$ in F_{n+k+1}, which is impossible since $[h(c_{s_0u0}), h(c_{s_0u1})] \in [F_{n+k+1}, F_{n+k+1}]$. Similarly we have $h(c_{s_0u1}) \in F_{n+k+1}$.

On the other hand, neither $h(c_{s_0u0})$ nor $h(c_{s_0u1})$ belongs to $[F_{n+k+1}, F_{n+k+1}]$ by Lemma 3.10. Hence at least one of $h(c_{s_0u00})$ and $h(c_{s_0u01})$ does not belong to F_{n+k+1} and consequently neither $h(c_{s_0u00})$ nor $h(c_{s_0u01})$ belongs to F_{n+k+1} by Lemma 2.7.

Hence $h(c_{s_0u0})$ is conjugate to d_t or d^-_t with $lh(t) = n + k + 1$ by Lemma 3.9. Similarly, $h(c_{s_0u1})$ is conjugate to $d_{t'}$ or $d^-_{t'}$ with $lh(t') = n+k+1$. Since $[h(c_{s_0u0}), h(c_{s_0u1})]$ is conjugate to $[d_{t_0v0}, d_{t_0v1}]$ or $[d_{t_0v0}, d_{t_0v1}]$ in F_{n+k+1}, $h(c_{s_0u0})$ and $h(c_{s_0u1})$ are conjugate to d_{t_0vj} or $d^-_{t_0vj}$ for some $j \in \underline{2}$ and for each $j \in \underline{2}$ the element d_{t0vj} is conjugate to exactly one of $h(c_{s_0u0})$, $h(c_{s_0u1})$, $h(c_{s_0u0})^-$ and $h(c_{s_0u1})^-$ by Lemma 3.11. Hence (1) and (2) hold for $k + 1$. Now we have shown the induction step and finished the proof. □

Remark 3.12. Though the conclusion of Theorem 3.1 is rather simple, embeddings from G^{S_0} into G^S may be complicated. In particular automorphisms on G^{S_0} may be complicated, since the following hold:

$$[dc^-d^-, dcd^-c^-d^-] = dc^-d^-dcd^-c^-d^-dcd^-dcdc^-d^- = cdc^-d^- = [c, d].$$

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