3. Optimal quadratic G^0 approximant

In the quadratic G^0 case control points determining the approximant p are given as

```
In[ ]:= B0[u_] = (1 - u)^2; B1[u_] = 2 (1 - u) u; B2[u_] = u^2;
        b0 = {c, -√(1 - c^2)}; b1 = {ξ, 0};
        b2 = {c, √(1 - c^2)};
        b20[t_, c_, ξ_] = Sum[bj Bj[(t + 1)/2], {j, 0, 2}];
        ψ20[t_, c_, ξ_] = b20[t, c, ξ]〚1〛^2 + b20[t, c, ξ]〚2〛^2 - 1 // FullSimplify;
```

where $\xi > 0$. The function $\psi$ is a quartic polynomial depending only on the parameter ξ and can be expressed as

```
In[ ]:= ψ20[t, c, ξ] == 1/8 (1 - t^2) (2 (1 - t^2) ξ^2 + 4 c (1 + t^2) ξ - 8 + 2 c^2 (1 - t^2)) // Simplify
Out[ ]= True
```

It increases on $(0, \infty)$ for all $t \in (-1, 1)$ and $0 \leq c < 1$. Thus, the optimal inner approximant is uniquely determined by the maximal $\xi > 0$ for which $\max_{t \in (-1,1)} \psi(t, \xi) = 0$ and the optimal outer approximant is also uniquely given by the minimal $\xi > 0$ for which $\min_{t \in (-1,1)} \psi(t, \xi) = 0$. Let us denote them as $\xi^{\vee}$ and $\xi^{\wedge}$, respectively.

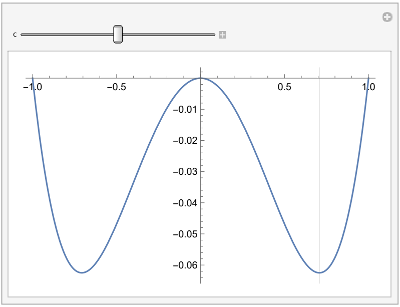
Since $\psi(\cdot, \xi)$ is symmetric on $(-1, 1)$, the parameter $\xi^{\vee}$ is obtained as a solution of the equation $\psi(0, \xi^{\vee}) = 0$, i.e., $\xi^{\vee} = 2 - c$ and the parameter $\xi^{\wedge}$ is determined as a solution of the equation $\psi'(1, \xi) = 0$, i.e., $\xi^{\wedge} = \frac{1}{c}$. Note that in the latter case we get the G^1 approximant.

Optimal inner quadratic G^0 approximant

```
In[ ]:= Solve[ψ20[0, c, ξ] == 0, ξ]
Out[ ]= {{ξ → -2 - c}, {ξ → 2 - c}}
```

```
In[ ]:= ξ20i[c_] = 2 - c;
        Manipulate[Plot[ψ20[t, c, ξ20i[c]], {t, -1, 1}, GridLines → {{√2/2}, {}}], {{c, 1/2}, 0, 1}]
```

Out[ ]=



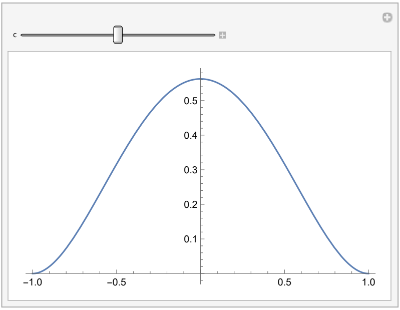
The calculation of the optimal parameter and the error:

```
In[∘]:= cc = √2/2;

        Print[ "The optimal parameter is ξ˅=", ξ20i[cc], " and the error is ", Simplify[ -ψ20[ √2/2, cc, ξ20i[cc]]],
          "=", N[ -ψ20[ √2/2, cc, ξ20i[cc]]], "."]

        The optimal parameter is ξ˅=2 - 1/√2  and the error is  1/8 (3 - 2 √2) =0.0214466.
```

Optimal outer quadratic G^0 approximant

```
In[∘]:= Solve[D[ψ20[t, c, ξ], t] == 0 /. t → 1, ξ]

Out[∘]= {{ξ → 1/c}}
```

```
In[∘]:= ξ20o[c_] = 1/c;

        Manipulate[Plot[ψ20[t, c, ξ20o[c]], {t, -1, 1}], {{c, 1/2}, 0, 1}]

Out[∘]=
```



The calculation of the optimal parameter and the error:

```
In[∘]:= cc = √2/2;

        Print["The optimal parameter is ξ^=", ξ20o[cc], " and the error is ", Simplify[ψ20[0, cc, ξ20o[cc]]],
          "=", N[ψ20[0, cc, ξ20o[cc]]], "."]

        The optimal parameter is ξ^=√2 and the error is 1/8=0.125.
```

4. Optimal cubic G^1 approximant

In this case, the control points of an approximant p are given as

```
In[∘]:= B₀[u_] = (1 - u)³; B₁[u_] = 3 (1 - u)² u; B₂[u_] = 3 (1 - u) u²; B₃[u_] = u³;
        b₀ = {c, - √(1 - c²)}; b₁ = b₀ + ξ {√(1 - c²) , c};
        b₃ = {c, √(1 - c²)}; b₂ = b₃ + ξ {√(1 - c²) , -c};
        b31[t_, c_, ξ_] = Sum[bⱼ Bⱼ[(t + 1)/2], {j, 0, 3}];
        ψ31[t_, c_, ξ_] = b31[t, c, ξ]⟦1⟧² + b31[t, c, ξ]⟦2⟧² - 1 // FullSimplify;
```

where $\xi > 0$. Note that the G^1 condition at the boundary points has already been incorporated. The error function $\psi$ is a polynomial of degree 6, which can be written as

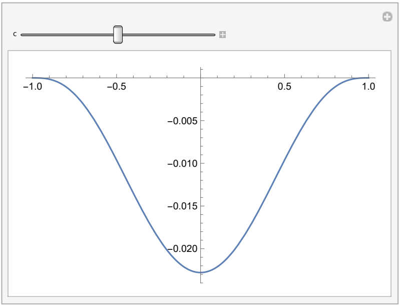
```
In[∘]:= ψ31[t, c, ξ] == 1/16 (1 - t^2)^2 (9 ξ^2 (t^2 c^2 + 1 - c^2) + 12 ξ (2 - t^2) √(1 - c^2) c - 4 (4 - t^2) (1 - c^2)) // Simplify

Out[∘]= True
```

It is again clear that $\psi(t,\cdot)$ is an increasing function for all $t \in (-1, 1)$ and $0 \le c < 1$. Since in addition $\psi(\cdot, \xi)$ is a polynomial of degree six with a positive leading coefficient, the parameter $\xi^\wedge$ of the optimal outer approximant must fulfil the condition $\psi(0, \xi^\wedge) = 0$, and consequently $\xi^\wedge = \frac{4s}{3(1+c)}$.

On the other hand, the condition for the approximant being an inner one is that $t = \pm 1$ are triple zeros of $\psi$, i.e., $\psi''(\pm 1, \xi) = 0$ which implies the solution $\xi^\vee = \frac{2}{3} s \left(\sqrt{3+c^2} - c\right)$.

Optimal inner cubic G^1 approximant

```
In[∘]:= Solve[D[ψ31[t, c, ξ], {t, 2}] == 0 /. t → 1, ξ] // Simplify

Out[∘]= {{ξ → -2/3 (c √(1 - c^2) + √(3 - 2 c^2 - c^4))}, {ξ → 2/3 (-c √(1 - c^2) + √(3 - 2 c^2 - c^4))}}

In[∘]:= Assuming[0 < c < 1, 2/3 (-c √(1 - c^2) + √(3 - 2 c^2 - c^4)) == 2/3 √(1 - c^2) (√(3 + c^2) - c) // Simplify]

Out[∘]= True

In[∘]:= ξ31i[c_] = 2/3 √(1 - c^2) (√(3 + c^2) - c);
        Manipulate[Plot[ψ31[t, c, ξ31i[c]], {t, -1, 1}], {{c, 1/2}, 0, 1}]

Out[∘]=
```

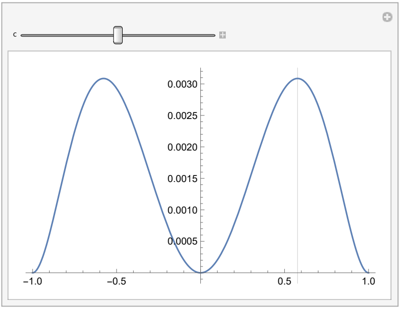


The calculation of the optimal parameter and the error:

```
In[∘]:= cc = √2/2;
        Print["The optimal parameter is ξ^v=", Simplify[ξ31i[cc]], " and the error is ",
         Simplify[-ψ31[0, cc, ξ31i[cc]]], "=", N[-ψ31[0, cc, ξ31i[cc]]], "."]

        The optimal parameter is ξ^v=1/3 (-1 + √7) and the error is 1/16 (8 - 3 √7)=0.00392163.
```

Optimal outer cubic G^1 approximant

```
In[∘]:= Solve[ψ31[0, c, ξ] == 0, ξ] // Simplify

Out[∘]= {{ξ → 4 √(1 - c^2)/(3 + 3 c)}, {ξ → 4 √(1 - c^2)/(3 (-1 + c))}}

In[∘]:= ξ31o[c_] = 4 √(1 - c^2)/(3 (1 + c));

        Manipulate[Plot[ψ31[t, c, ξ31o[c]], {t, -1, 1}, GridLines → {{√3/3}, {}}], {{c, 1/2}, 0, 1}]

Out[∘]=
```



The calculation of the optimal parameter and the error:

```
In[∘]:= cc = √2/2;
        Print["The optimal parameter is ξ^=", Simplify[ξ31o[cc]], " and the error is ",
         Simplify[ψ31[√3/3, cc, ξ31o[cc]]], "=", N[ψ31[√3/3, cc, ξ31o[cc]]], "."]

        The optimal parameter is ξ^=4/3 (-1 + √2) and the error is 17/54 - 2 √2/9=0.000545134.
```

5. Optimal quartic G^2 approximant

As we will see in the following, the analysis gets much more complicated here as in the previous two cases. Considering G^1 and G^2 conditions, we can express control points of an approximant p as

```
In[∘]:= B0[u_] = (1 - u)^4; B1[u_] = 4 (1 - u)^3 u; B2[u_] = 6 (1 - u)^2 u^2; B3[u_] = 4 (1 - u) u^3; B4[u_] = u^4;
        b0 = {c, -√(1 - c^2)}; b1 = b0 + ξ {√(1 - c^2), c}; b2 = {(3 - 4 ξ^2)/(3 c), 0};
        b4 = {c, √(1 - c^2)}; b3 = b4 + ξ {√(1 - c^2), -c};
        b42[t_, c_, ξ_] = Sum[bj Bj[(t + 1)/2], {j, 0, 4}];
        ψ42[t_, c_, ξ_] = b42[t, c, ξ][[1]]^2 + b42[t, c, ξ][[2]]^2 - 1 // FullSimplify;
```

where again $\xi > 0$. The free parameter ξ was chosen similarly as in the previous section. The control point b_2 is not defined for $c = 0$, but this case can be obtained as the limit $c \to 0$. The function $\psi$ is now a polynomial of degree 8, and with

```
In[∘]:= α1 = -1/2 √(1 - c^2) (√(3 + c^2) + c);
        α2 = 1/2 √(1 - c^2) (√(3 + c^2) - c);
```

$$\beta 1=\frac{1}{2}\left(c\sqrt{1-c^2}-\sqrt{3+8c+6c^2-c^4}\right);$$

$$\beta 2=\frac{1}{2}\left(c\sqrt{1-c^2}-\sqrt{3-8c+6c^2-c^4}\right);$$

$$\beta 3=\frac{1}{2}\left(c\sqrt{1-c^2}+\sqrt{3-8c+6c^2-c^4}\right);$$

$$\beta 4=\frac{1}{2}\left(c\sqrt{1-c^2}+\sqrt{3+8c+6c^2-c^4}\right);$$

it can be written as

In[∘]:= $$\psi 42[t,c,\xi]==\frac{\left(t^2-1\right)^3}{64\,c^2}\left(16\,(\xi-\alpha 1)^2\,(\xi-\alpha 2)^2\,t^2-16\,(\xi-\beta 1)\,(\xi-\beta 2)\,(\xi-\beta 3)\,(\xi-\beta 4)\right)\ \text{// Simplify}$$

Out[∘]= True

Unfortunately, $\psi(t,\,\cdot)$ is no longer a monotone function, and a more involved analysis is needed. Observe first that

In[∘]:= $$\psi 42[0,c,\xi]==\frac{1}{4\,c^2}\,(\xi-\beta 1)\,(\xi-\beta 2)\,(\xi-\beta 3)\,(\xi-\beta 4)\ \text{// Simplify}$$

Out[∘]= True

and

In[∘]:= $$\gamma 1=\frac{1}{2}\left(c\sqrt{1-c^2}-\sqrt{3+6c^2-c^4}\right);$$

$$\gamma 2=\frac{1}{2}\,c\sqrt{1-c^2};$$

$$\gamma 3=\frac{1}{2}\left(c\sqrt{1-c^2}+\sqrt{3+6c^2-c^4}\right);$$

$$\text{D}[\psi 42[0,c,\xi],\xi]==\frac{1}{c^2}\,(\xi-\gamma 1)\,(\xi-\gamma 2)\,(\xi-\gamma 3)\ \text{// Simplify}$$

Out[∘]= True

Let

In[∘]:= $$\text{td}[c\_,\xi\_]=\frac{48\sqrt{1-c^2}\,\xi^3}{c}+48\,\xi^2-12\sqrt{1-c^2}\left(\frac{3}{c}+c\right)\xi+12\left(1-c^2\right);$$

$$\text{td}[c,\xi]==(\text{D}[\psi 42[t,c,\xi],\{t,3\}]\ /.\ t\to 1)\ \text{// Simplify}$$

Out[∘]= True

Then

In[∘]:= $$\text{td}[c,0]==12\left(1-c^2\right)\ \text{// Simplify}$$

$$(\text{D}[\text{td}[c,\xi],\xi]\ /.\ \xi\to 0)==-\frac{12}{c}\sqrt{1-c^2}\left(3+c^2\right)\ \text{// Simplify}$$

$$\text{Assuming}\left[0<c<1,\ \text{td}[c,\beta 3]==12\,(1-c)^4\left((2+c)-\sqrt{1+c}\,\sqrt{3+c}\right)^2\ \text{// Simplify}\right]$$

Out[∘]= True

Out[∘]= True

Out[∘]= True

Furthermore,

In[∘]:= $$\delta 1=\frac{-2\,c-\sqrt{9-2\,c^2-3\,c^4}}{6\sqrt{1-c^2}};$$

$$\delta2 = \frac{-2\,c + \sqrt{9 - 2\,c^2 - 3\,c^4}}{6\,\sqrt{1 - c^2}};$$

$$\texttt{D[td[c, ξ], ξ]} == \frac{144}{c}\,\sqrt{1 - c^2}\,(\xi - \delta1)\,(\xi - \delta2)\ \texttt{// Simplify}$$

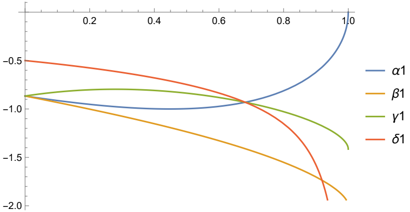
$$\texttt{td[c, δ2]} == -\frac{48\,(1 - c^2)^3\,(27 - 18\,c^2 - c^4)}{c\left(8\,c\,\left(27\,(1 - c^2)^2 + 22\,c^2 - 18\,c^4\right) + 4\,(9 - 2\,c^2 - 3\,c^4)\,\sqrt{9 - 2\,c^2 - 3\,c^4}\right)}\ \texttt{// Simplify}$$

```
Out[∘]= True

Out[∘]= True
```

It is easy to see

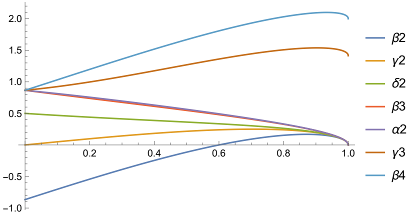
```
In[∘]:= Plot[{α1, β1, γ1, δ1}, {c, 0, 1}, PlotLegends → "Expressions"]
```

Out[∘]=



```
In[∘]:= Plot[{β2, γ2, δ2, β3, α2, γ3, β4}, {c, 0, 1}, PlotLegends → "Expressions"]
```

Out[∘]=



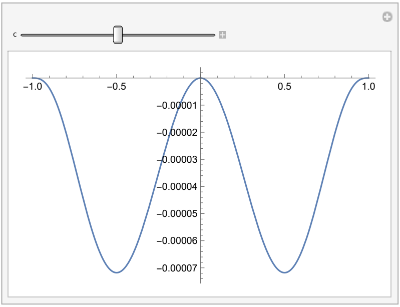
Optimal inner quartic G^2 approximant

For

$$\texttt{In[∘]:= ξ42i[c\_]} = \frac{1}{2}\left(c\,\sqrt{1 - c^2} + \sqrt{3 - 8\,c + 6\,c^2 - c^4}\right);$$

$$\texttt{Manipulate}\left[\texttt{Plot[ψ42[t, c, ξ42i[c]], \{t, -1, 1\}]}, \left\{\left\{c, \frac{1}{2}\right\}, 0, 1\right\}\right]$$

Out[∘]=



we have

```
In[∘]:= Assuming[0 < c < 1, ψ42[t, c, ξ42i[c]] == -1/4 (1 - c)^4 (2 + c - Sqrt[3 + 4 c + c^2])^2 t^2 (1 - t^2)^3 // Simplify]

Out[∘]= True
```

A parameter $\xi > \beta_4$ does not induce an inner approximant since $\psi(0, \xi) > 0$. Suppose $\beta_3 < \xi \le \beta_4$. Since $t\,d$ is increasing on $[\beta_3, \beta_4]$ there is $\epsilon > 0$, such that $\psi(t, \xi) < \psi(t, \xi^\vee)$ for $t \in (1-\epsilon, 1)$. If ξ induces a better approximant as $\xi^\vee$, then the graphs of $\psi(\cdot, \xi)$ and $\psi(\cdot, \xi^\vee)$ have two intersection on $(0, 1)$, therefore ten intersections on $[-1, 1]$, which is not possible. If $c < \frac{3}{5}$, there are no $\xi \in [0, \xi^\vee]$ which induce an inner approximant.

Suppose $c \ge \frac{3}{5}$ and $0 \le \xi \le \beta_2$. Since $l\,c(\xi^\vee) < l\,c(\beta_2)$, the parameter $\xi^\vee$ induces a better approximant then β_2. Since $t\,d$ is decreasing on $[0, \beta_2]$ a parameter $\xi < \beta_2$ induces a worse approximant then β_2. Therefore, $\xi^\vee$ induces the optimal inner approximant. In the limit case, when $c = 0$, we get $\xi^\vee = \frac{\sqrt{3}}{2}$ and the abscissa of the control point b_2 is

$$\lim_{c\to 0} \frac{3 - 4\beta_3^2}{3c} = \frac{2}{3}\left(4 - \sqrt{3}\right).$$

The calculation of the optimal parameter and the error:

```
In[∘]:= cc = Sqrt[2]/2;
Print["The Optimal parameter is ξ∨=", Simplify[ξ42i[cc]], " and the error is ",
 Simplify[-ψ42[1/2, cc, ξ42i[cc]]], "=", N[-ψ42[1/2, cc, ξ42i[cc]]], "."]

The Optimal parameter is ξ∨=1/4 (1 + Sqrt[23 - 16 Sqrt[2]]) and the error is
 -(27 (-5 Sqrt[23 - 16 Sqrt[2]] + 4 (-10 + 7 Sqrt[2] + Sqrt[46 - 32 Sqrt[2]])))/4096 =7.11383×10^-6.
```

The case c=0

```
In[∘]:= ξξ = Limit[(3 - 4 ξ42i[c]^2)/(3 c), c → 0] // Simplify
Print["The optimal parameter is ξ∨=", Simplify[ξξ], " and the error is ",
 Simplify[-Limit[ψ42[1/2, c, ξ42i[c]], c → 0]], "=", N[-Limit[ψ42[1/2, c, ξ42i[c]], c → 0]], "."]
```

Out[∘]= $-\frac{2}{3}\left(-4+\sqrt{3}\right)$

The optimal parameter is ξ˅=$-\frac{2}{3}\left(-4+\sqrt{3}\right)$ and the error is $-\frac{27\left(-7+4\sqrt{3}\right)}{1024}$=0.00189308.

Optimal outer quartic G^2 approximant

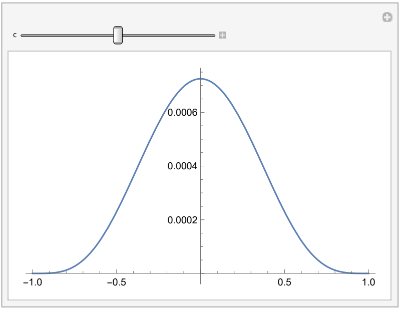
Let ξ^ be the only zero of $t\,d$ on the interval $[\delta_2, \infty)$. By the above calculations ξ^ < β_3, so ξ^ induces an outer approximant. If ξ > ξ^, then $t\,d\,(\xi) > 0$, hence ξ does not induce an outer approximant. Suppose that 0 ≤ ξ < ξ^. If $t\,d\,(\xi) < 0$, then ξ does not induce an outer approximant. Let $t\,d\,(\xi) \geq 0$. Since $l\,d\,(\xi) > l\,d\,(\xi^\wedge)$ the graphs of $\psi(\cdot, \xi)$ and $\psi(\cdot, \xi^\wedge)$ intersect on $(1, \infty)$. Hence, the graphs can not have an intersection on the interval $(-1, 1)$, so ξ induces a worse approximant as ξ^.

In the limit case, when $c = 0$, we get ξ^ = $\frac{\sqrt{3}}{2}$ and the abscissa of the control point b_2 is $\frac{8\sqrt{3}}{9}$.

In[∘]:= $$\text{ξ42o[c\_]} = \frac{\sqrt{9-2c^2-3c^4}}{3\sqrt{1-c^2}}\ \text{Cos}\left[\frac{1}{3}\ \text{ArcCos}\left[\frac{2c\left(27-32c^2+9c^4\right)}{\left(-9+2c^2+3c^4\right)\sqrt{9-2c^2-3c^4}}\right]\right] - \frac{c}{3\sqrt{1-c^2}};$$

$$\text{Manipulate}\left[\text{Plot[ψ42[t, c, ξ42o[c]], \{t, -1, 1\}]}, \left\{\left\{c, \frac{1}{2}\right\}, 0, 1\right\}\right]$$

Out[∘]=



The calculation of the optimal parameter and the error:

In[∘]:= $$\text{cc} = \frac{\sqrt{2}}{2};$$

$$\text{Print}\left[\text{"The optimal parameter is ξ^=", N[ξ42o[cc]], " and the error is ", N}\left[\text{ψ42}\left[\frac{\sqrt{3}}{3}, \text{cc, ξ42o[cc]}\right]\right], \text{"."}\right]$$

```
The optimal parameter is ξ^=0.402437 and the error is 0.0000138132.
```

The case c=0

In[∘]:= $$\text{ξξ} = \text{Limit}\left[\frac{3-4\,\text{ξ42o[c]}^2}{3c}, c \to 0\right] \text{ // Simplify}$$

$$\text{Print}\left[\text{"The optimal parameter is ξ˅=", Simplify[ξξ], " and the error is ",}\right.$$
$$\left.\text{Simplify}\left[\text{Limit}\left[\text{ψ42}\left[\frac{1}{2}, \text{c, ξ42o[c]}\right], c \to 0\right]\right], \text{"=", N}\left[\text{Limit}\left[\text{ψ42}\left[\frac{1}{2}, \text{c, ξ42o[c]}\right], c \to 0\right]\right], \text{"."}\right]$$

Out[∘]= $\frac{8}{3\sqrt{3}}$

The optimal parameter is $\xi^{\vee}=\frac{8}{3\sqrt{3}}$ and the error is $\frac{27}{4096}=0.0065918$.

6. Optimal cubic G^0 approximant

This section deals with cubic one-sided approximants of c. The control points determining the approximant p are

```
In[]:= B0[u_] = (1 - u)^3; B1[u_] = 3 (1 - u)^2 u; B2[u_] = 3 (1 - u) u^2; B3[u_] = u^3;
       b0 = {c, -√(1 - c^2)}; b1 = {ξ, -η};
       b2 = {ξ, η}; b3 = {c, √(1 - c^2)};
       b30[t_, c_, ξ_, η_] = Sum[bj Bj[(t + 1)/2], {j, 0, 3}];
       ψ30[t_, c_, ξ_, η_] = b30[t, c, ξ, η][[1]]^2 + b30[t, c, ξ, η][[2]]^2 - 1 // FullSimplify;
```

where $\xi>0$. The function \$\psi\$ is of degree 6 and it is of the form

```
In[]:= ψ30[t, c, ξ, η] == 1/16 (t^2 - 1) ((3 η - √(1 - c^2))^2 t^4 + (16 (1 - c^2) - 9 (η + √(1 - c^2))^2 + 9 (ξ - c)^2) t^2 + (16 - (3 ξ + c)^2)) //
        Simplify
Out[]= True
```

hence

```
In[]:= lc[c_, ξ_, η_] = 1/16 (3 η - √(1 - c^2))^2;
       lc[c, ξ, η] == Coefficient[ψ30[t, c, ξ, η], t^6] // Simplify
       ψ30[0, c, ξ, η] == (1/16 (3 ξ + c)^2 - 1) // Simplify
       (D[ψ30[t, c, ξ, η], t] /. t → 1) == 3 (1 - ξ c - η √(1 - c^2)) // Simplify
Out[]= True
Out[]= True
Out[]= True
```

We deal with the two-parametric optimization problem here, and the analysis is much more involved than in the previous cases. Let us consider the inner and the outer approximation separately again.

Optimal inner cubic G^0 approximant

Following the idea of equioscillation of the best polynomial approximant in the functional case, we might expect equioscillation of $\psi$. Since it is an inner approximant, it should touch the abscissa from below as many times as possible. Thus, the guess is that its graph looks like the one in Figure 1 (left). Since $\psi$ is of degree 6 and it has two simple zeros at ±1, the remaining two zeros on $(-1,1)$ should be double ones and related to the maxima of the Chebyshev polynomial of degree six. Consequently, they equal to $\pm\frac{1}{2}$ and we must have

$$\psi\left(\frac{1}{2},\xi,\eta\right)=0,\quad \psi'\left(\frac{1}{2},\xi,\eta\right)=0.$$

It will be shown that the above system has a unique solution with $\xi>c$, and that this solution implies the optimal approximant. Observe that (3) is an algebraic system of equations for ξ and η. Two polynomials of a Gröbner basis for the ordering $\eta\prec\xi$ are $p_1(\cdot)=(\cdot-c)\,g_1(\cdot)$ and g_2, where

```
In[]:= g1[ξ_] = 2187 ξ^3 + 3807 c ξ^2 + (1953 c^2 - 3840) ξ + 245 c^3 - 4352 c;
```

$$\texttt{g2}[\xi\_,\ \eta\_] = -8\ (1-c^2)\ \eta + \sqrt{1-c^2}\ \ (27\ \xi^2 + 10\ c\ \xi + 3\ c^2 - 40);$$

In[∘]:= $$\texttt{GroebnerBasis}\left[\left\{\psi30\left[\frac{1}{2},\ c,\ \xi,\ \eta\right],\ \left(\texttt{D}[\psi30[t,\ c,\ \xi,\ \eta],\ t]\ \texttt{/.}\ t \to \frac{1}{2}\right)\right\},\ \{\eta,\ \xi\}\right][\![1\ ;;\ 2]\!] == \{(\xi - c)\ \texttt{g1}[\xi],\ \texttt{g2}[\xi,\ \eta]\}\ \texttt{//}\ \texttt{Simplify}$$

Out[∘]= True

The zero $\xi = c$ of p_1 is not admissible, since the corresponding polynomial approximant reduces to the line segment, which definitely is not the optimal one. Thus, we are restricted to the positive zeros of g_1. Let $I = (\xi_{min},\ \xi_{max})$, where

In[∘]:= $$\xi_{\texttt{min}} = \frac{1}{27}\left(-5\ c + 8\ \sqrt{2}\ \ \sqrt{9-c^2}\right);$$

$$\xi_{\texttt{max}} = \frac{4-c}{3};$$

Note that $c < 1 < \xi_{min} < \xi_{max}$ for all $c \in [0,\ 1)$.

Lemma 1. The only positive zero of g_1 is in I.

Proof. We have

In[∘]:= $$\texttt{g1}\left[-\frac{37}{27}\ c\right] == \frac{8192}{9}\ c\ \left(1-c^2\right)\ \texttt{//}\ \texttt{Simplify}$$

$$\texttt{g1}[0] == c\ \left(245\ c^2 - 4352\right)\ \texttt{//}\ \texttt{Simplify}$$

$$\texttt{g1}[\xi_{\texttt{min}}] == -\frac{2048\ \left(1-c^2\right)^2}{\left(4\ c + \sqrt{2}\ \ \sqrt{9-c^2}\right)}\ \texttt{//}\ \texttt{Simplify}$$

$$\texttt{g1}[\xi_{\texttt{max}}] == 64\ (1-c)^3\ \texttt{//}\ \texttt{Simplify}$$

Out[∘]= True

Out[∘]= True

Out[∘]= True

Out[∘]= True

Since the leading coefficient of g_1 is positive, g_1 has two non-positive zeros and the unique positive one in I. This completes the proof. ∎

Since $g_2(\cdot,\ \eta)$ is a linear polynomial, the unique solution of the system (3) is $(\xi^{\vee},\ \eta^{\vee})$, where $\xi^{\vee}$ is the unique zero of g_1 in I, and

$$\eta^{\vee} = \frac{27\ \xi^{\vee 2} + 10\ c\ \xi^{\vee} + 3\ c^2 - 40}{8\ \sqrt{1-c^2}}.$$

Let $J = (\eta_{min},\ \eta_{max})$, where

In[∘]:= $$\eta_{\texttt{min}} = \frac{\sqrt{1-c^2}}{3};$$

$$\eta_{\texttt{max}} = \frac{3-c}{1+c}\ \frac{\sqrt{1-c^2}}{3};$$

Lemma 2. If $(\xi^{\vee},\ \eta^{\vee})$ is the solution of the system (3), then $\eta^{\vee} \in J$.

Proof. Let

In[∘]:= $$\texttt{f}[\xi\_] = \frac{27\ \xi^2 + 10\ c\ \xi + 3\ c^2 - 40}{8\ \sqrt{1-c^2}};$$

Then

In[∘]:= $$\texttt{D}[\texttt{f}[\xi],\ \xi] == \frac{27\ \xi + 5\ c}{4\ \sqrt{1-c^2}}\ \texttt{//}\ \texttt{Simplify}$$

Out[∘]= True

and f is strictly increasing on I. Thus

In[∘]:=

$$\texttt{f}[\xi_{\min}] == \frac{\sqrt{1-c^2}}{3}\ \texttt{// Simplify}$$

$$\texttt{f}[\xi_{\max}] == \frac{3-c}{1+c}\ \frac{\sqrt{1-c^2}}{3}\ \texttt{// Simplify}$$

Out[∘]= `True`

Out[∘]= `True`

Since $\xi^{\vee} \in I$ and $\eta^{\vee} = f(\xi^{\vee})$ by (6), the proof is complete. ∎

We shall see in the following that $(\xi^{\vee}, \eta^{\vee})$ induces the polynomial approximant p with

$$(\xi^{\vee}, \eta^{\vee}) = \underset{(.,\%)}{\operatorname{argmin}}\left(\max_{t\in[-1,1]} |\psi(t, \xi, \eta)|\right),$$

i.e., the optimal one.

Let us suppose that (ξ, η) induces a better approximant. If $\xi < \xi^{\vee}$, then by (2) we have $\psi(0, \xi, \eta) \le \psi(0, \xi^{\vee}, \eta^{\vee}) < 0$.

Thus $|\psi(0, \xi, \eta)| \ge |\psi(0, \xi^{\vee}, \eta^{\vee})|$, and $\psi(\cdot, \xi, \eta)$ does not provide a smaller error than $\psi(\cdot, \xi^{\vee}, \eta^{\vee})$.

If $\xi \ge \xi^{\vee}$ and $\eta > \eta^{\vee}$, then by (2)

$\psi'(1, \xi, \eta) < \psi'(1, \xi^{\vee}, \eta^{\vee})$ and $\operatorname{lc}(\psi(\cdot, \xi, \eta)) > \operatorname{lc}(\psi(\cdot, \xi^{\vee}, \eta^{\vee}))$.

Consequently, $\psi(\cdot, \xi, \eta)$ and $\psi(\cdot, \xi^{\vee}, \eta^{\vee})$ must intersect six times in $[-1, 1]$ (see Figure 2 (left)) and at least twice out of $[-1, 1]$, which contradicts the fact that $\psi(\cdot, \xi, \eta)$ is of degree six. If $\xi = \xi^{\vee}$ and $\eta < \eta^{\vee}$, then $\psi'(1, \xi, \eta) > \psi'(1, \xi^{\vee}, \eta^{\vee})$, so $\psi(\cdot, \xi, \eta)$ and $\psi(\cdot, \xi^{\vee}, \eta^{\vee})$ must intersect at least eight times in $[-1, 1]$ (see Figure 2 (left) again), which can not be the case.

Thus, we are left with $\xi > \xi^{\vee}$ and $\eta \le \eta^{\vee}$. The case where $\psi'(1, \xi, \eta) \ge \psi'(1, \xi^{\vee}, \eta^{\vee})$ is eliminated by using the same argument as in the previous case, and the following lemma eliminates the last case, i.e., $\psi'(1, \xi, \eta) < \psi'(1, \xi^{\vee}, \eta^{\vee})$.

Lemma 3. If $\xi > \xi^{\vee}$, $\eta \le \eta^{\vee}$ and $\psi'(1, \xi, \eta) < \psi'(1, \xi^{\vee}, \eta^{\vee})$, then $\psi\left(\frac{1}{2}, \xi, \eta\right) > 0$ and $\psi(\cdot, \xi, \eta)$ is not an inner approximant.

Proof. Since the pair of parameters (ξ, η) provides an inner approximant, we have $\xi \le \xi_{max}$. The assumptions of the lemma then imply

$$\xi \in (\xi^{\vee}, \xi_{\max}], \quad \eta \in \left(-\frac{c}{\sqrt{1-c^2}}\,\xi + \frac{c}{\sqrt{1-c^2}}\,\xi^{\vee} + \eta^{\vee}, \eta^{\vee}\right].$$

Thus $(\xi, \eta) \in T$, where T is the triangle given by (7) (see Figure 3 (left)). We need to show that $\psi\left(\frac{1}{2}, \xi, \eta\right) > 0$ for $(\xi, \eta) \in T$, or, equivalently, that T lies outside of the ellipse E given by $\psi\left(\frac{1}{2}, \xi, \eta\right) = 0$. The implicit equation of the ellipse reads

In[∘]:=

$$\psi30\left[\frac{1}{2}, c, \xi, \eta\right] == \frac{1}{1024}\left(4\,(9\,\xi + 7\,c)^2 + \left(9\,\eta + 13\,\sqrt{1-c^2}\right)^2 - 1024\right)\ \texttt{// Simplify}$$

Out[∘]= `True`

revealing that its centre is in the third quadrant. Thus, it is concave and decreasing in the first quadrant. The triangle T is above the line l_T given by the equation

$$\eta = -\frac{c}{\sqrt{1-c^2}}\,\xi + \frac{c}{\sqrt{1-c^2}}\,\xi^{\vee} + \eta^{\vee}.$$

Let l_E\$ be the tangent line of the ellipse E at $(\xi^{\vee}, \eta^{\vee})$. Note that the lines l_T and l_E intersect in $(\xi^{\vee}, \eta^{\vee})$ (see Figure 3 (left)). The result of the lemma will follow if we prove that the slope of l_T is greater than the slope of l_E since then the triangle T is out of the ellipse E. Both slopes can be easily computed using (8) and (9) and we have to verify the inequality

$$\sqrt{1-c^2}\,(12\,\xi^{\vee} + 5\,c) - 3\,c\,\eta^{\vee} > 0.$$

But it follows from the fact that $\sqrt{1-c^2}\,(12\,\xi + 5\,c) - 3\,c\,\eta > 0$ for $(\xi, \eta) \in I \times J$ (we simply check the inequality at the corner points of $I \times J$) and the fact that $(\xi^{\vee}, \eta^{\vee}) \in I \times J$. ∎

Let us summarize the results in the following theorem.

Theorem 4. The optimal inner cubic G^0 approximant of a circular arc is given by the pair of parameters $(\xi^{\vee}, \eta^{\vee}) \in I \times J$, where

$\xi^\vee$ is a unique zero of g_1 in I and $\eta^\vee$ is a unique zero of $g_2(\cdot, \xi^\vee)$ in J.

The calculation of the optimal parameter and the error:

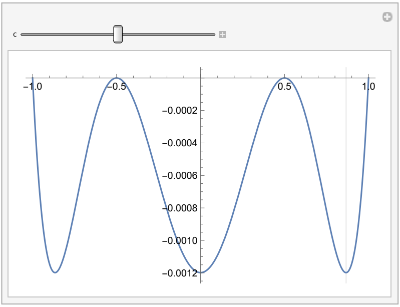
In[∘]:=

$$\xi30\mathrm{i}[c\_] = \frac{32\sqrt{15+c^2}}{81}\,\mathrm{Cos}\Big[\frac{1}{3}\,\mathrm{ArcCos}\Big[\frac{c\,(63+c^2)}{(15+c^2)^{3/2}}\Big]\Big] - \frac{47\,c}{81};$$

$$\eta30\mathrm{i}[c\_] = \frac{27\,\xi30\mathrm{i}[c]^2 + 10\,c\,\xi30\mathrm{i}[c] + 3\,c^2 - 40}{8\sqrt{1-c^2}};$$

$$\mathrm{Manipulate}\Big[\mathrm{Plot}\Big[\psi30[t, c, \xi30\mathrm{i}[c], \eta30\mathrm{i}[c]], \{t, -1, 1\}, \mathrm{GridLines} \to \Big\{\Big\{\frac{\sqrt{3}}{2}\Big\}, \{\}\Big\}\Big], \Big\{\Big\{c, \frac{1}{2}\Big\}, 0, 1\Big\}\Big]$$

Out[∘]=



In[∘]:=

$$\mathrm{cc} = \frac{\sqrt{2}}{2};$$

$$\mathrm{Print}\Big[\text{"The pair of optimal parameters is }\{\xi^\vee,\eta^\vee\}\text{=", }\{\mathrm{N}[\xi30\mathrm{i}[\mathrm{cc}]], \mathrm{N}[\eta30\mathrm{i}[\mathrm{cc}]]\}, \text{" and the error is ", } \mathrm{N}\Big[-\psi30\Big[\frac{\sqrt{3}}{2}, \mathrm{cc}, \xi30\mathrm{i}[\mathrm{cc}], \eta30\mathrm{i}[\mathrm{cc}]\Big]\Big], \text{"."}\Big]$$

```
The pair of optimal parameters is {ξ˅,η˅}={1.09748, 0.31486} and the error is 0.000220288.
```

Optimal outer cubic G^0 approximant

We will show that the optimal outer approximant is G^2 approximant. From the equations $\psi(0, \xi^\wedge, \eta^\wedge) = 0$ and $\psi'(1, \xi^\wedge, \eta^\wedge) = 0$ we get

In[∘]:=

$$\xi30\mathrm{o}[c\_] = \frac{4-c}{3};$$

$$\eta30\mathrm{o}[c\_] = \frac{3-4\,c+c^2}{3\sqrt{1-c^2}};$$

Suppose that a pair (ξ, η) induces a better outer approximant. From $\psi(0, \xi, \eta) \geq \psi(0, \xi^\wedge, \eta^\wedge) = 0$ we get $\xi \geq \xi^\wedge$. From $1 - \sqrt{1-c^2}\;\eta - c\,\xi = \frac{1}{3}\frac{d}{dt}\psi(1, \xi, \eta) \leq \frac{1}{3}\frac{d}{dt}\psi(1, \xi^\wedge, \eta^\wedge) = 1 - \sqrt{1-c^2}\;\eta^\wedge - c\,\xi^\wedge = 0$ we get $\xi \geq \frac{1}{c}\left(\sqrt{1-c^2}\;(\eta^\wedge - \eta) + c\,\xi^\wedge\right)$. From

In[∘]:=

$$\psi30\Big[\frac{\sqrt{3}}{3}, c, \xi, \eta\Big] == \frac{1}{108}\left(\left(3\,\eta + 5\sqrt{1-c^2}\right)^2 + 27\,(\xi + c)^2 - 108\right) \mathbin{//} \mathrm{Simplify}$$

Out[∘]= True

we get $\eta \le \eta\hat{}$. From the last two inequalities we get

$$0 \le \left(3\,\eta\hat{} + 5\,\sqrt{1-c^2}\right)^2 + 27\,(\xi\hat{} + c)^2 - \left(3\,\eta + 5\,\sqrt{1-c^2}\right)^2 - 27\,(\xi + c)^2$$
$$\ge \left(3\,\eta\hat{} + 5\,\sqrt{1-c^2}\right)^2 + 27\,(\xi\hat{} + c)^2 - \left(3\,\eta + 5\,\sqrt{1-c^2}\right)^2 - 27\left(\frac{1}{c}\left(\sqrt{1-c^2}\,(\eta\hat{} - \eta) + c\,\xi\hat{}\right) + c\right)^2$$
$$= -\frac{3}{c^2}\,(\eta\hat{} - \eta)\left((9 - 6\,c^2)\,(\eta\hat{} - \eta) + 2\,c^2\left(4\,\sqrt{1-c^2} - 3\,\eta\hat{}\right) + 18\,c\,\sqrt{1-c^2}\,\xi\hat{}\right) \ge 0.$$

The last inequality follows since $\eta \le \eta\hat{} \le \frac{4}{3}\sqrt{1-c^2}$. Hence all inequalites are equalities, so $(\xi, \eta) = (\xi\hat{}, \eta\hat{})$.

The calculation of the optimal parameter and the error:

In[∘]:=
$$\texttt{cc} = \frac{\sqrt{2}}{2};$$
$$\texttt{Print}\left[\texttt{"The pair of optimal parameter is \{}\xi\hat{}\texttt{,}\eta\hat{}\texttt{\}=", \{N[}\xi\texttt{30o[cc]], N[}\eta\texttt{30o[cc]]\}, " and the error is ",}\right.$$
$$\left.\texttt{N}\left[\psi\texttt{30}\left[\frac{\sqrt{3}}{3}\texttt{, cc, }\xi\texttt{30o[cc], }\eta\texttt{30o[cc]}\right]\right]\texttt{, "."}\right]$$

```
The pair of optimal parameter is {ξ^,η^}={1.09763, 0.316582} and the error is 0.000545134.
```

---

7. Optimal quartic G^1 approximant

In the G^1 quartic case the control points of an approximant p are given as

In[∘]:=
$$\texttt{B}_0[u\_] = (1-u)^4;\ \texttt{B}_1[u\_] = 4\,(1-u)^3\,u;\ \texttt{B}_2[u\_] = 6\,(1-u)^2\,u^2;\ \texttt{B}_3[u\_] = 4\,(1-u)\,u^3;\ \texttt{B}_4[u\_] = u^4;$$
$$\texttt{b}_0 = \left\{c, -\sqrt{1-c^2}\right\};\ \texttt{b}_1 = \left\{c, -\sqrt{1-c^2}\right\} + \xi\left\{\sqrt{1-c^2}, c\right\};\ \texttt{b}_2 = \{\eta, 0\};$$
$$\texttt{b}_3 = \left\{c, \sqrt{1-c^2}\right\} + \xi\left\{\sqrt{1-c^2}, -c\right\};\ \texttt{b}_4 = \left\{c, \sqrt{1-c^2}\right\};$$
$$\texttt{b41}[t\_, c\_, \xi\_, \eta\_] = \texttt{Sum}\left[\texttt{b}_j\,\texttt{B}_j\left[\frac{t+1}{2}\right], \{j, 0, 4\}\right];$$
$$\psi\texttt{41}[t\_, c\_, \xi\_, \eta\_] = \texttt{b41}[t, c, \xi, \eta][\![1]\!]^2 + \texttt{b41}[t, c, \xi, \eta][\![2]\!]^2 - 1\ \texttt{// FullSimplify};$$

where $\xi > 0$ due to the G^1 continuity. The function $\psi$ is again a polynomial of degree 8 and can be written as

In[∘]:=
$$\texttt{lc}[c\_, \xi\_, \eta\_] = \frac{1}{64}\left(3\,\eta - 4\,\sqrt{1-c^2}\,\xi - 3\,c\right)^2;$$
$$\texttt{a1}[c\_, \xi\_, \eta\_] = \frac{1}{64}\left(32\,c^2\,\xi^2 + 32\,\xi^2 - 64\,c\,\sqrt{1-c^2}\,\xi - 18\,\eta^2 + 36\,c\,\eta - 34\,c^2 + 16\right);$$
$$\texttt{a0}[c\_, \xi\_, \eta\_] = \frac{1}{64}\left(\left(3\,\eta + 4\,\sqrt{1-c^2}\,\xi + 5\,c\right)^2 - 64\right);$$
$$\psi\texttt{41}[t, c, \xi, \eta] == (t^2-1)^2\,(\texttt{lc}[c, \xi, \eta]\,t^4 + \texttt{a1}[c, \xi, \eta]\,t^2 + \texttt{a0}[c, \xi, \eta])\ \texttt{// Simplify}$$
$$(\texttt{D}[\psi\texttt{41}[t, c, \xi, \eta], \{t, 2\}]\ \texttt{/.}\ t \to 1) == 8\,\xi^2 + 6\,c\,\eta - 6\ \texttt{// Simplify}$$

```
Out[∘]= True

Out[∘]= True
```

Optimal inner quartic G^1 approximant

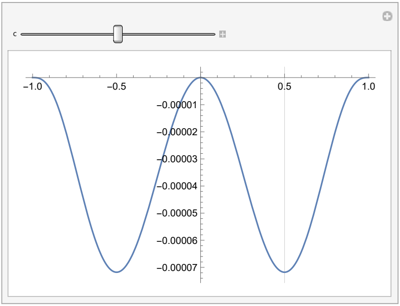
It will turn out that the optimal inner quartic G^1 approximant is actually the optimal G^2 approximant from Section 5.1. Recall that, in this case, we have $\psi\,(0, \xi, \eta) = 0$ and $\frac{d^2\,\psi}{d\,t^2}\,(1, \xi, \eta) = 0$, and the appropriate solution which induces the best approximant is given by

In[∘]:=
$$\xi\texttt{41i}[c\_] = \frac{c\,(1-c^2) + (1-c)^2\,\sqrt{3 + 4\,c + c^2}}{2\,\sqrt{1-c^2}};$$

```
η41i[c_] = 1/3 (8 - 7 c + 2 c^3 - 2 (1 - c)^2 Sqrt[3 + 4 c + c^2]);
```

```
In[∘]:= Manipulate[Plot[ψ41[t, c, ξ41i[c], η41i[c]], {t, -1, 1}, GridLines → {{1/2}, {}}], {{c, 1/2}, 0, 1}]
```

Out[∘]=



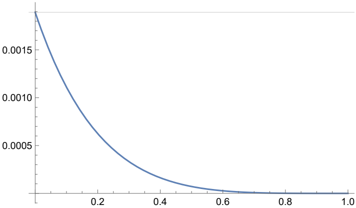
with the error

```
In[∘]:= -ψ41[1/2, c, ξ41i[c], η41i[c]] == 1/1024 (27 (1 - c)^4 ((2 + c) - Sqrt[3 + 4 c + c^2])^2) // Simplify
```

```
Out[∘]= True
```

```
In[∘]:= Plot[1/1024 (27 (1 - c)^4 ((2 + c) - Sqrt[3 + 4 c + c^2])^2), {c, 0, 1}, GridLines → {{}, {27 (2 - Sqrt[3])^2/1024}}]
```

Out[∘]=



Suppose that a pair $(\xi, \eta) \neq (\xi^\vee, \eta^\vee)$ induces the optimal inner G^1 approximant. By Section 5.1, we know that the induced approximant is not a G^2 approximant. First, we show that $\eta \geq 0$.

Suppose that $\eta < 0$. Since $\psi(0, \xi, \eta) \geq \min_{t\in[-1,1]} \psi(\cdot, \xi^\vee, \eta^\vee) =: m$, we have $3\eta + 4\sqrt{1-c^2}\,\xi + 5c < -8\sqrt{1+m} < -2$. Then

$$\mathrm{lc}(\psi(\cdot, \xi, \eta)) = \frac{1}{64}\left(-3\eta + 4\sqrt{1-c^2}\,\xi + 3c\right)^2 \geq \frac{1}{64}\left(8\sqrt{1-c^2}\,\xi + 8c + 2\right)^2 \geq \frac{1}{64}(8c+2)^2$$

$$> \frac{1}{4}(1-c)^4\left(7 + 8c + 2c^2 - 2(2+c)\sqrt{3+4c+c^2}\right) = \mathrm{lc}(\psi(\cdot, \xi^\vee, \eta^\vee)).$$

Together with the inequality $\psi''(1, \xi, \eta) < \psi''(1, \xi^\vee, \eta^\vee) = 0$ we get that $\psi(t, \xi, \eta) < \psi(t, \xi^\vee, \eta^\vee) = 0$ for all t close but not equal to 1. Hence the graphs of $\psi(\cdot, \xi, \eta)$ and $\psi(\cdot, \xi^\vee, \eta^\vee)$ have one intersection on $(1, \infty)$ and one on $(-\infty, -1)$. Since they have at least double zeros at ± 1, they have at most two intersections on $(-1, 1)$, so there are no intersections on $(-1, 1)$. Hence (ξ, η) induces a worse G^1 inner approximant then $(\xi^\vee, \eta^\vee)$, hence $\eta \geq 0$.

Since $\eta \geq 0$ and $\psi(t, \xi, \eta) = b_2\eta^2 + b_1\eta + b_0$, where

```
In[∘]:= b2 = 9/64 (1 - t^2)^4;
        b1 = 3/32 (1 - t^2)^2 (c (5 + 6 t^2 - 3 t^4) + 4 Sqrt[1 - c^2] (1 - t^4) ξ);
        ψ41[t, c, ξ, η] == b2 η^2 + b1 η + ψ41[t, c, ξ, 0] // Simplify
Out[∘]= True
```

for all $t \in (-1, 1)$, the function $\psi(t, \cdot, \eta)$ is increasing. If $\psi(0, \xi, \eta) < 0$ we can enlarge ξ and get better inner G^1 approximant. Hence $\psi(0, \xi, \eta) = 0$ and therefore $\eta = \eta(\xi) = \frac{1}{3}\left(8 - 4\sqrt{1-c^2}\,\xi - 5c\right)$. If $\xi > \xi^{\vee}\$$ then $\psi(0, \xi, \eta(\xi)) > 0$, so $(\xi, \eta(\xi))$ does not induce an inner approximant. If $\xi < \xi^{\vee}$, then $\psi\left(\frac{1}{2}, \xi, \eta(\xi)\right) < \psi\left(\frac{1}{2}, \xi^{\vee}, \eta^{\vee}\right) = \min_{t\in[-1,1]} \psi(\cdot, \xi^{\vee}, \eta^{\vee})$, hence (ξ, η) induces worse approximant then $(\xi^{\vee}, \eta^{\vee})$.

Optimal outer quartic G^1 approximant

In order to simplify the coefficients of the polynomial $\psi$, we introduce two new variables
$u = 3\eta - 4\sqrt{1-c^2}\,\xi, \quad v = 3\eta + 4\sqrt{1-c^2}\,\xi$
which implies $\xi = \frac{v-u}{8\sqrt{1-c^2}}$ and $\eta = \frac{1}{6}(u+v)$. Now $\psi$ can be written as $\psi_{uv}(t, u, v) = \psi(t, \xi(u, v), \eta(u, v))$ and we get

```
In[∘]:= ψuv[t_, c_, u_, v_] = ψ41[t, c, (v - u)/(8 Sqrt[1 - c^2]), 1/6 (u + v)];
        Coefficient[ψuv[t, c, u, v], t^8] == 1/64 (u - 3 c)^2 // Simplify
        ψuv[0, c, u, v] == 1/64 ((v + 5 c)^2 - 64) // Simplify
        (D[ψuv[t, c, u, v], {t, 2}] /. t → 1) == (v - u)^2/(8 (1 - c^2)) + c (v + u) - 6 // Simplify
Out[∘]= True
Out[∘]= True
Out[∘]= True
```

Observe that $\xi > 0$ due to the G^1 interpolation condition and $\psi_{uv}(0, u, v) \geq 0$ since the approximant is the outer one. This implies
$v > u$ and $|v + 5c| \geq 8$.
Let us first assume that $v \geq 8 - 5c$.
It is clear that ψ_{uv} is symmetric and has two double zeros at ±1. Following the idea of equioscillation of the best polynomial approximant in the functional case, we might expect that ψ_{uv} equioscillates. Since it is an outer approximant, it may touch the abscissa several times. Thus, the guess is that its graph looks like on Figure 1 (right). But then its four double zeros on $[-1, 1]$ must be related to the minima of the Chebyshev polynomial of degree eight. Consequently, the two double zeros on $(-1, 1)$ are equal to $1 - \sqrt{2}$ and $\sqrt{2} - 1$. Since these points are also minima of ψ_{uv}, we must have
$\psi_{uv}\left(\sqrt{2} - 1, u, v\right) = 0, \quad \psi'_{uv}\left(\sqrt{2} - 1, u, v\right) = 0.$
This leads to the system of two algebraic equations for u and v. Its Gröbner basis according to the ordering $v \prec u$ are polynomials p_1 and p_2, given as
$$p_1(u) = \frac{1}{1-c^2}\left(\alpha_4 u^4 + \alpha_3 u^3 + \alpha_2 u^2 + \alpha_1 u + \alpha_0\right),$$
where

```
In[∘]:= α4 = c^2 (c^2 + 1);
        α3 = -8 (3 + √2) c^5 + 4 (4 + 5 √2) c^3 + 4 (4 + 3 √2) c;
        α2 = 2 (125 + 68 √2) c^6 - 2 (229 + 182 √2) c^4 + 4 (140 + 97 √2) c^2 - 4 (65 + 46 √2);
        α1 = -88 (14 + 9 √2) c^7 + 12 (116 + 89 √2) c^5 - 84 (16 + 11 √2) c^3 + 4 (50 + 36 √2) c;
        α0 = 5 (457 + 312 √2) c^8 - (1655 + 1236 √2) c^6 + 12 (16 + 9 √2) c^4 + 4 (163 + 114 √2) c^2 + 272 + 192 √2;
        p1[u_] = 1/(1 - c^2) (α4 u^4 + α3 u^3 + α2 u^2 + α1 u^1 + α0);
```

and

$$p_2(u, v) = \frac{1}{c^2 - 1}(\beta_1 v + \beta_0(u)),$$

where

```
In[∘]:= β1 = -2 (3 + 2 √2 + 2 c^2 - (1 + 6 √2) c^4 + 4 (4 + √2) c^6 + 4 c^8);
        β0[u_] = 36 (3 - 2 √2) c^9 + 16 (3 √2 - 5) c^8 u + (329 - 198 √2 + (12 - 8 √2) u^2) c^7 + (142 √2 - 253) c^6 u +
           (169 - 198 √2 + (55 - 34 √2) u^2) c^5 + (174 √2 - 51 + (2 √2 - 3) u^2) c^4 u + (4 (43 + 9 √2) - 5 (6 √2 - 7) u^2) c^3 -
           (16 (7 + 3 √2) u + (3 - 2 √2) u^3) c^2 + 2 (23 + 16 √2) u - 4 c (10 + 9 √2 + √2 u^2);
        p2[u_, v_] = 1/(c^2 - 1) (β1 v + β0[u]);
        GroebnerBasis[{Simplify[ψuv[√2 - 1, c, u, v]], Simplify[(D[ψuv[t, c, u, v], t] /. t → √2 - 1)]}, {v, u}] ==
          {p1[u], p2[u, v]} // Simplify
Out[∘]= True
```

Let us define the domain of interest for (u, v) as

$$D = (3c, \tilde{u}) \times [8 - 5c, \infty),$$

where

```
In[∘]:= ũ = k c^2 - (2 k - 3) c + k /. k → 2 √((2 √2 - 1)/7);
```

Note that D is a subset of (11). We shall prove in the following that there exists a unique solution $(u^\wedge, v^\wedge) \in D$ of (12) with $u^\wedge$ the unique zero of p_1 on $I_{41} = (3c, \tilde{u})$ and $v^\wedge = -\frac{\beta_0(u^\wedge)}{\beta_1} \geq 8 - 5c$. Moreover, this solution is *optimal*, i.e., it induces the polynomial approximant p with $\psi_{u^\wedge v^\wedge}$ for which

$$(u^\wedge, v^\wedge) = \underset{(u,v)}{\operatorname{argmin}}\left(\max_{t\in[-1,1]} |\psi_{\text{uv}}(t, u, v)|\right).$$

Let us first confirm the following lemma.

Lemma 5. The polynomial p_1 has precisely one zero $u^\wedge$ on I_{41}, and the system (12) has precisely one solution $(u^\wedge, v^\wedge)$ in D.

Proof. By (13) and (14) the leading coefficient of p_1 is

```
In[∘]:= Coefficient[p1[u], u^4] == (c^2 + c^4)/(1 - c^2) // Simplify
Out[∘]= True
```

thus $\lim_{u\to\pm\infty} p_1(u) = \infty$. A computer algebra system reveals that

```
In[∘]:= a = {-4 (11757 + 8314 √2), -20 (20970 + 14827 √2), -5 (300175 + 212256 √2), -2868170 - 2028204 √2,
          -10 (328607 + 232384 √2), -8 (301424 + 213155 √2), -16 (71127 + 50296 √2), -1152 (239 + 169 √2)};
        (1 + c) p1[-6 √2 - 5] == Total[Table[a⟦i + 1⟧ c^i (1 - c)^(7-i), {i, 0, 7}]] // Simplify
        b = {272 + 192 √2, 1740 + 1224 √2, 36 (105 + 73 √2), 9 (197 + 124 √2), 18 (97 + 60 √2)};
        (1 - c^2) p1[0] == Total[Table[b⟦i + 1⟧ c^(2 i) (1 - c^2)^(4-i), {i, 0, 4}]] // Simplify
        p1[3 c] == 16 (17 + 12 √2) (1 - c^2)^3 // Simplify
        d = {-(28 (468177 + 331052 √2))/(196 (17 + 12 √2) + 1174 √(-7 + 14 √2) + 830 √(-14 + 28 √2)),
          -(1023648947 + 723832272 √2)/(30571 + 21584 √2 + 10630 √(-7 + 14 √2) + 7510 √(-14 + 28 √2)),
          -(44 (334596323 + 236596644 √2))/(116750 + 82480 √2 + 40512 √(-7 + 14 √2) + 28610 √(-14 + 28 √2)),
          -(8 (8319080395 + 5882517894 √2))/(84728 √(-7 + 14 √2) + 59822 √(-14 + 28 √2) + 4 (60467 + 42733 √2)),
          -(4 (6933599848 + 4902830366 √2))/(73675 + 52080 √2 + 26381 √(-7 + 14 √2) + 18624 √(-14 + 28 √2)),
          -(224 (301412347 + 213132210 √2))/(78508 √(-7 + 14 √2) + 55420 √(-14 + 28 √2) + 28 (7577 + 5356 √2)),
          -(112 (1009965 + 714158 √2))/(750 + 530 √2 + 289 √(-7 + 14 √2) + 204 √(-14 + 28 √2)),
          -(28 (550063 + 388956 √2))/(507 + 358 √2 + 204 √(-7 + 14 √2) + 144 √(-14 + 28 √2))};
        49 (1 + c) p1[ũ] == 16 c (1 - c)^3 Total[Table[d⟦i + 1⟧ c^i (1 - c)^(7-i), {i, 0, 7}]] // Simplify
Out[∘]= True
Out[∘]= True
Out[∘]= True
Out[∘]= True
```

Thus $p_1\left(-6\sqrt{2}-5\right) < 0$, $p_1(0) > 0$, $p_1(3c) > 0$, and $p_1(\tilde{u}) < 0$ and the result of the lemma follows since p_1 is quartic. ∎

Let us confirm that the solution $(u^{\wedge}, v^{\wedge})$ induces the optimal approximant. If not, then there exists a pair $(u, v) \in D$, $(u, v) \neq (u^{\wedge}, v^{\wedge})$, for which

$$\max_{t\in[-1,1]} |\psi_{uv}(t, u, v)| < \max_{t\in[-1,1]} |\psi_{uv}(t, u^{\wedge}, v^{\wedge})|.$$

If $v > v^{\wedge}$, then $\psi_{uv}(0, \cdot, v) > \psi_{uv}(0, \cdot, v^{\wedge})$ since $v > 8 - 5c$ by (11), and (u, v) can not induce better approximant than $(u^{\wedge}, v^{\wedge})$. Thus $v \leq v^{\wedge}$. Consider first the case $u > u^{\wedge}$. By Lemma 5 $u^{\wedge} > 3c$ and (10) implies lc $(\psi_{uv}(t, u, v)) >$ lc $(\psi_{uv}(t, u^{\wedge}, v^{\wedge}))$. If $\psi''_{uv}(1, u^{\wedge}, v^{\wedge}) > \psi''_{uv}(1, u, v)$, then the graphs of $\psi_{uv}(\cdot, u^{\wedge}, v^{\wedge})$ and $\psi_{uv}(\cdot, u, v)$ intersect on $(1, \infty)$. Moreover, if (u, v) induces a better approximant as $(u^{\wedge}, v^{\wedge})$, then the graphs of $\psi_{uv}(\cdot, u^{\wedge}, v^{\wedge})$ and $\psi_{uv}(\cdot, u, v)$ would have to intersect at least eight times on $[-1, 1]$, which is not possible since $\psi_{uv}(\cdot, u, v)$ is a polynomial of degree at most 8 (see Figure 2 (right)). Similarly, if $\psi''_{uv}(1, u^{\wedge}, v^{\wedge}) \leq \psi''_{uv}(1, u, v)$, the above-mentioned graphs would have to intersect at least ten times on $[-1, 1]$ which is again a contradiction. Finally, let us consider the case $v \leq v^{\wedge}$ and $u \leq u^{\wedge}$. If $\psi''_{uv}(1, u^{\wedge}, v^{\wedge}) \leq \psi''_{uv}(1, u, v)$, then the graphs of $\psi_{uv}(\cdot, u^{\wedge}, v^{\wedge})$ and $\psi_{uv}(\cdot, u, v)$ would have to intersect at least ten times on $[-1, 1]$ which is again an obvious contradiction. So, we are left with the case $\psi''_{uv}(1, u^{\wedge}, v^{\wedge}) > \psi''_{uv}(1, u, v)$. The following lemma reveals that, in this case, the approximant is not an outer one.

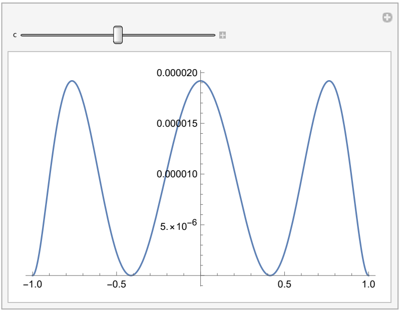
In[∘]:=

$$\texttt{u41o[c\_]} = -\frac{4\sqrt{249+176\sqrt{2}-\left(217+152\sqrt{2}\right)c^2+2c^4}}{3c}\,\texttt{Cos}\left[\frac{1}{3}\texttt{ArcCos}\left[\frac{\left(5555+3928\sqrt{2}-3\left(2423+1713\sqrt{2}\right)c^2+3\left(478+337\sqrt{2}\right)c^4+2\sqrt{2}\,c^6\right)}{\left(249+176\sqrt{2}-\left(217+152\sqrt{2}\right)c^2+2c^4\right)^{3/2}}\right]-\frac{4\pi}{3}\right]+\frac{-22-16\sqrt{2}+15c^2+4\sqrt{2}\,c^2}{3c};$$

$$\texttt{v41o[c\_]} = -\frac{\beta_0[\texttt{u41o[c]}]}{\beta_1};$$

$$\texttt{Manipulate}\left[\texttt{Plot[}\psi\texttt{uv[t, c, u41o[c], v41o[c]], \{t, -1, 1\}]}, \left\{\left\{\texttt{c}, \frac{1}{2}\right\}, 0, 1\right\}\right]$$

Out[∘]=



Lemma 6. Let $v \le v^\wedge$ and $u \le u^\wedge$. If $\psi''_{uv}(1, u^\wedge, v^\wedge) > \psi''_{uv}(1, u, v)$, then $\psi_{uv}\left(\sqrt{2}-1, u, v\right) < 0$.

Proof. Let us define

In[∘]:=

$$\texttt{f[u\_, v\_] = (D[}\psi\texttt{uv[t, c, u, v], \{t, 2\}] - D[}\psi\texttt{uv[t, c, u41o[c], v41o[c]], \{t, 2\}] /. t} \to \texttt{1);}$$

$$\texttt{g[u\_, v\_] = }\psi\texttt{uv}\left[\sqrt{2}-1, \texttt{c, u, v}\right];$$

Observe that $f(u, v) = 0$ is a parabola

In[∘]:=

$$\texttt{Det}\left[\left\{\left\{\texttt{Coefficient}\left[\texttt{f[u, v]}, \texttt{u}^2\right], \frac{1}{2}\texttt{Coefficient[f[u, v], u v]}\right\}, \left\{\frac{1}{2}\texttt{Coefficient[f[u, v], u v]}, \texttt{Coefficient}\left[\texttt{f[u, v]}, \texttt{v}^2\right]\right\}\right\}\right] == 0 \texttt{ // Simplify}$$

Out[∘]= True

and $g(u, v) = 0$ is an ellipse (see Figure 3, right)

In[∘]:=

$$\texttt{Det}\left[\left\{\left\{\texttt{Coefficient}\left[\texttt{g[u, v]}, \texttt{u}^2\right], \frac{1}{2}\texttt{Coefficient[g[u, v], u v]}\right\}, \left\{\frac{1}{2}\texttt{Coefficient[g[u, v], u v]}, \texttt{Coefficient}\left[\texttt{g[u, v]}, \texttt{v}^2\right]\right\}\right\}\right] == \frac{\left(577-408\sqrt{2}\right)c^2}{64\left(1-c^2\right)} \texttt{ // Simplify}$$

Out[∘]= True

with leading coefficients of their tangent lines at (u, v) equal to

In[∘]:=

$$\texttt{cp[u\_, v\_]} = \frac{4\,c^3 - 4\,c - u + v}{-4\,c^3 + 4\,c - u + v};$$

$$\texttt{ce[u\_, v\_]} = \frac{\left(12\,\sqrt{2} - 17\right) u + 2\,c\,\left(15 - 11\,\sqrt{2} + \left(11\,\sqrt{2} - 15\right) c^2 + \left(7 - 5\,\sqrt{2}\right) c\,u\right) + \left(3 - 2\,\sqrt{2}\right) v}{\left(2\,\sqrt{2} - 3\right) u + v + 2\,c\,\left(1 + \sqrt{2} - c\,\left(\left(1 + \sqrt{2}\right) c + \left(\sqrt{2} - 1\right) v\right)\right)};$$

$$-\frac{\texttt{D[f[u, v], u]}}{\texttt{D[f[u, v], v]}} == \texttt{cp[u, v] // Simplify}$$

$$-\frac{\texttt{D[g[u, v], u]}}{\texttt{D[g[u, v], v]}} == \texttt{ce[u, v] // Simplify}$$

Out[∘]= True

Out[∘]= True

respectively. With the help of a computer algebra system, it can be again shown that $c_p(u, v), c_e(u, v) > 0$

In[∘]:=

$$\texttt{cp1} = \left\{\left\{\{8, 9\}, \left\{8 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}, 9 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\}, \left\{\{12, 15\}, \left\{12 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}, 15 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\}, \{\{0, 3\}, \{0, 3\}\}, \{\{0, 1\}, \{0, 1\}\}\right\};$$

$$\texttt{cp2} = \left\{\left\{\{8, 9\}, \left\{8 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}, 9 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\}, \left\{\{20, 23\}, \left\{20 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}, 23 - 2\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\}, \{\{16, 19\}, \{16, 19\}\}, \{\{0, 1\}, \{0, 1\}\}\right\};$$

```
And @@ Map[# ≥ 0 &, Flatten[Join[cp1, cp2]]]
```

$$\texttt{cp}\left[(1 - u)\,3\,c + u\,\tilde{u}, v + 8 - 5\,c\right] ==$$
$$\texttt{Total}\left[\texttt{Flatten}\left[\texttt{Table}\left[\texttt{cp1}[\![i + 1, j + 1, k + 1]\!]\,(1 - c)^{3-i}\,c^i\,(1 - u)^{1-j}\,u^j\,(1 - v)^{1-k}\,v^k, \{i, 0, 3\}, \{j, 0, 1\}, \{k, 0, 1\}\right]\right]\right] /$$
$$\texttt{Total}\left[\texttt{Flatten}\left[\texttt{Table}\left[\texttt{cp2}[\![i + 1, j + 1, k + 1]\!]\,(1 - c)^{3-i}\,c^i\,(1 - u)^{1-j}\,u^j\,(1 - v)^{1-k}\,v^k, \{i, 0, 3\}, \{j, 0, 1\}, \{k, 0, 1\}\right]\right]\right]\ \texttt{// Simplify}$$

Out[∘]= True

Out[∘]= True

In[∘]:=

$$\texttt{ce1} =$$
$$\left\{\left\{\left\{24 - 16\,\sqrt{2}, 27 - 18\,\sqrt{2}\right\}, \left\{24 - 16\,\sqrt{2} + \frac{2}{7}\left(-17 + 12\,\sqrt{2}\right)\sqrt{-7 + 14\,\sqrt{2}}, 27 - 18\,\sqrt{2} + \frac{2}{7}\left(-17 + 12\,\sqrt{2}\right)\sqrt{-7 + 14\,\sqrt{2}}\right\}\right\},\right.$$
$$\left\{\left\{60 - 40\,\sqrt{2}, 72 - 48\,\sqrt{2}\right\}, \left\{60 - 40\,\sqrt{2} - 68\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)} + 48\,\sqrt{\frac{2}{7}\left(-1 + 2\,\sqrt{2}\right)}, 72 - 48\,\sqrt{2} - 68\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)} + 48\,\sqrt{\frac{2}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\},$$
$$\left\{\left\{36 - 24\,\sqrt{2}, 54 - 36\,\sqrt{2}\right\}, \left\{\frac{2}{7}\left(-3 + 2\,\sqrt{2}\right)\left(-42 + \sqrt{-7 + 14\,\sqrt{2}}\right), \frac{2}{7}\left(-3 + 2\,\sqrt{2}\right)\left(-63 + \sqrt{-7 + 14\,\sqrt{2}}\right)\right\}\right\},$$
$$\left.\left\{\left\{0, 12 - 8\,\sqrt{2}\right\}, \left\{0, 12 - 8\,\sqrt{2}\right\}\right\}, \left\{\left\{0, 3 - 2\,\sqrt{2}\right\}, \left\{0, 3 - 2\,\sqrt{2}\right\}\right\}\right\};$$

$$\texttt{ce2} = \left\{\left\{\{8, 9\}, \left\{8 + \frac{2}{7}\left(-3 + 2\,\sqrt{2}\right)\sqrt{-7 + 14\,\sqrt{2}}, 9 + \frac{2}{7}\left(-3 + 2\,\sqrt{2}\right)\sqrt{-7 + 14\,\sqrt{2}}\right\}\right\},\right.$$
$$\left\{\left\{20 + 8\,\sqrt{2}, 8\left(3 + \sqrt{2}\right)\right\}, \left\{20 + 8\,\sqrt{2} - 12\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)} + 8\,\sqrt{\frac{2}{7}\left(-1 + 2\,\sqrt{2}\right)}, 8\left(3 + \sqrt{2}\right) - 12\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)} + 8\,\sqrt{\frac{2}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\},$$
$$\left\{\left\{28 + 8\,\sqrt{2}, 6\left(6 + \sqrt{2}\right)\right\}, \left\{28 + 8\,\sqrt{2} - 6\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)} + 4\,\sqrt{\frac{2}{7}\left(-1 + 2\,\sqrt{2}\right)}, 6\left(6 + \sqrt{2}\right) - 6\,\sqrt{\frac{1}{7}\left(-1 + 2\,\sqrt{2}\right)} + 4\,\sqrt{\frac{2}{7}\left(-1 + 2\,\sqrt{2}\right)}\right\}\right\}, \left\{\left\{16, -4\left(-6 + \sqrt{2}\right)\right\}, \left\{16, -4\left(-6 + \sqrt{2}\right)\right\}\right\},$$
$$\left.\left\{\left\{0, 3 - 2\,\sqrt{2}\right\}, \left\{0, 3 - 2\,\sqrt{2}\right\}\right\}\right\};$$

```
And @@ Map[# ≥ 0 &, Flatten[Join[ce1, ce2]]]
ce[(1 - u) 3 c + u ũ, v + 8 - 5 c] ==
  Total[Flatten[Table[ce1[[i + 1, j + 1, k + 1]] (1 - c)^(4-i) c^i (1 - u)^(1-j) u^j (1 - v)^(1-k) v^k, {i, 0, 4}, {j, 0, 1}, {k, 0, 1}]]]/
   Total[Flatten[Table[ce2[[i + 1, j + 1, k + 1]] (1 - c)^(4-i) c^i (1 - u)^(1-j) u^j (1 - v)^(1-k) v^k, {i, 0, 4}, {j, 0, 1},
      {k, 0, 1}]]] // Simplify
```

```
Out[∘]= True

Out[∘]= True
```

and $c_p(u, v) > c_e(u, v)$ for $(u, v) \in D$.

In[∘]:=

$$
\begin{aligned}
&\texttt{cc} = \Big\{\Big\{\Big\{128\left(-1+\sqrt{2}\right),\ 288\left(-1+\sqrt{2}\right),\ 162\left(-1+\sqrt{2}\right)\Big\},\\
&\quad\Big\{-\frac{64}{7}\left(28-28\sqrt{2}-4\sqrt{-7+14\sqrt{2}}+3\sqrt{-14+28\sqrt{2}}\right),\ -\frac{8}{7}\left(-504\left(-1+\sqrt{2}\right)-68\sqrt{-7+14\sqrt{2}}+51\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{36}{7}\left(63-63\sqrt{2}-8\sqrt{-7+14\sqrt{2}}+6\sqrt{-14+28\sqrt{2}}\right)\Big\},\\
&\quad\Big\{-\frac{8}{7}\left(85-93\sqrt{2}-32\sqrt{-7+14\sqrt{2}}+24\sqrt{-14+28\sqrt{2}}\right),\ -\frac{8}{7}\left(198-214\sqrt{2}-68\sqrt{-7+14\sqrt{2}}+51\sqrt{-14+28\sqrt{2}}\right),\\
&\quad \frac{1}{7}\left(-918+982\sqrt{2}+288\sqrt{-7+14\sqrt{2}}-216\sqrt{-14+28\sqrt{2}}\right)\Big\}\Big\},\\
&\Big\{\Big\{704\left(-1+\sqrt{2}\right),\ 1608\left(-1+\sqrt{2}\right),\ 918\left(-1+\sqrt{2}\right)\Big\},\\
&\quad\Big\{-\frac{16}{7}\left(616-616\sqrt{2}-73\sqrt{-7+14\sqrt{2}}+55\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{8}{7}\left(-2814\left(-1+\sqrt{2}\right)-312\sqrt{-7+14\sqrt{2}}+235\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{4}{7}\left(-3213\left(-1+\sqrt{2}\right)-332\sqrt{-7+14\sqrt{2}}+250\sqrt{-14+28\sqrt{2}}\right)\Big\},\\
&\quad\Big\{-\frac{8}{7}\left(535-559\sqrt{2}-146\sqrt{-7+14\sqrt{2}}+110\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{8}{7}\left(1245-1293\sqrt{2}-312\sqrt{-7+14\sqrt{2}}+235\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{2}{7}\left(2889-2985\sqrt{2}-664\sqrt{-7+14\sqrt{2}}+500\sqrt{-14+28\sqrt{2}}\right)\Big\}\Big\},\\
&\Big\{\Big\{1408\left(-1+\sqrt{2}\right),\ 3312\left(-1+\sqrt{2}\right),\ 1944\left(-1+\sqrt{2}\right)\Big\},\\
&\quad\Big\{-\frac{64}{7}\left(-29\sqrt{-7+14\sqrt{2}}+22\left(14-14\sqrt{2}+\sqrt{-14+28\sqrt{2}}\right)\right),\\
&\quad -\frac{8}{7}\left(-5796\left(-1+\sqrt{2}\right)-500\sqrt{-7+14\sqrt{2}}+379\sqrt{-14+28\sqrt{2}}\right),\\
&\quad 3888\left(-1+\sqrt{2}\right)+2144\sqrt{\frac{1}{7}\left(-1+2\sqrt{2}\right)}-232\sqrt{-14+28\sqrt{2}}\Big\},\\
&\quad\Big\{-\frac{16}{7}\left(589-597\sqrt{2}-116\sqrt{-7+14\sqrt{2}}+88\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{8}{7}\left(2790-2822\sqrt{2}-500\sqrt{-7+14\sqrt{2}}+379\sqrt{-14+28\sqrt{2}}\right),\\
&\quad -\frac{8}{7}\left(1647-1663\sqrt{2}-268\sqrt{-7+14\sqrt{2}}+203\sqrt{-14+28\sqrt{2}}\right)\Big\}\Big\},\\
&\Big\{\Big\{1216\left(-1+\sqrt{2}\right),\ 3064\left(-1+\sqrt{2}\right),\ 1908\left(-1+\sqrt{2}\right)\Big\},\\
&\quad\Big\{-864\sqrt{\frac{2}{7}\left(-1+2\sqrt{2}\right)}+32\left(76\left(-1+\sqrt{2}\right)+5\sqrt{-7+14\sqrt{2}}\right),\\
&\quad 6128\left(-1+\sqrt{2}\right)-1896\sqrt{\frac{2}{7}\left(-1+2\sqrt{2}\right)}+352\sqrt{-7+14\sqrt{2}},\\
&\quad 3816\left(-1+\sqrt{2}\right)-1032\sqrt{\frac{2}{7}\left(-1+2\sqrt{2}\right)}+192\sqrt{-7+14\sqrt{2}}\Big\},\\
&\quad\Big\{-864\sqrt{\frac{2}{7}\left(-1+2\sqrt{2}\right)}+32\left(38\left(-1+\sqrt{2}\right)+5\sqrt{-7+14\sqrt{2}}\right),\\
&\quad 3064\left(-1+\sqrt{2}\right)-1896\sqrt{\frac{2}{7}\left(-1+2\sqrt{2}\right)}+352\sqrt{-7+14\sqrt{2}},
\end{aligned}
$$

```
    1908 (-1 + √2) - 1032 √(2/7 (-1 + 2 √2)) + 192 √(-7 + 14 √2)}},
  {{384 (-1 + √2), 1216 (-1 + √2), 882 (-1 + √2)},
   {-64/7 (-1 + √2) (-84 + √(-7 + 14 √2)), 2432 (-1 + √2) + 192 √(1/7 (-1 + 2 √2)) - 176 √(2/7 (-1 + 2 √2)),
    1764 (-1 + √2) + 128 √(1/7 (-1 + 2 √2)) - 16 √(-14 + 28 √2)},
   {-64/7 (-1 + √2) (-42 + √(-7 + 14 √2)), -16/7 (-532 (-1 + √2) - 12 √(-7 + 14 √2) + 11 √(-14 + 28 √2)),
    882 (-1 + √2) + 128 √(1/7 (-1 + 2 √2)) - 16 √(-14 + 28 √2)}},
  {{0, 176 (-1 + √2), 198 (-1 + √2)}, {0, 352 (-1 + √2), 396 (-1 + √2)}, {0, 176 (-1 + √2), 198 (-1 + √2)}},
  {{0, 32 (-1 + √2), 36 (-1 + √2)}, {0, 64 (-1 + √2), 72 (-1 + √2)}, {0, 32 (-1 + √2), 36 (-1 + √2)}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}};
And @@ Map[# ≥ 0 &, Flatten[cc]]
Simplify[
  ((4 c^3 - 4 c - u + v) ((2 √2 - 3) u + v + 2 c (1 + √2 - c ((1 + √2) c + (√2 - 1) v))) -
     ((12 √2 - 17) u + 2 c (15 - 11 √2 + (11 √2 - 15) c^2 + (7 - 5 √2) c u) + (3 - 2 √2) v) (-4 c^3 + 4 c - u + v)) /.
   {u → (1 - u) 3 c + u ũ, v → v + 8 - 5 c}] ==
 Total[Flatten[Table[cc⟦i + 1, j + 1, k + 1⟧ (1 - c)^(7-i) c^i (1 - u)^(2-j) u^j (1 - v)^(2-k) v^k, {i, 0, 7}, {j, 0, 2}, {k, 0, 2}]]] //
 Simplify
```

```
Out[∘]= True

Out[∘]= True
```

Thus, by (11) and Lemma 5, all these inequalities hold for $u = u^\wedge$ and $v = v^\wedge$. The parabola and the ellipse intersect at $(u^\wedge, v^\wedge)$, and $f(0, 0) = -\frac{1}{8(1-c^2)}(v^\wedge - u^\wedge)^2 - c(v^\wedge + u^\wedge) < 0$, $g(0, 0) = -\frac{16\sqrt{2} - 20 + (68\sqrt{2} - 97)c^2}{4} < 0$. Since in addition the ordinate of the intersection of tangent lines with v-axis is positive, the parabola and the ellipse are concave at $(u^\wedge, v^\wedge)$. This implies $\psi_{u\,v}(\sqrt{2} - 1, u, v) < 0$ and the result of the lemma is confirmed. Since, in addition, the ordinate of the intersection of the tangent line to ellipse with v-axis is positive, and the parabola is symmetric about the line $v = u$, the parabola and ellipse are concave at $(u^\wedge, v^\wedge)$. ∎

Finally, consider $v \le -8 - 5c$. By (11) we have $u < v$, therefore $u < -8 - 5c$, and $\mathrm{lc}(\psi_{u\,v}(\cdot, u, v)) = \frac{1}{64}(u - 3c)^2 \ge \frac{1}{64}(-8 - 5c - 3c)^2 = (1 + c)^2$. From the above calculation we have $\mathrm{lc}(\psi_{u\,v}(\cdot, u^\wedge, v^\wedge)) \le \frac{1}{64}(\tilde{u} - 3c)^2 = \frac{1}{112}(2\sqrt{2} - 1)(1 - c)^4 \le \mathrm{lc}(\psi_{u\,v}(\cdot, u, v))$. Hence, if the pair (u, v) induces a better outer approximant than the pair $(u^\wedge, v^\wedge)$, then the error graphs of induced approximants have at least ten intersections, which is impossible. We can now finally formulate the main result of this subsection.

Theorem 7. The optimal quartic G^1 outer approximant is given by the unique solution of the nonlinear algebraic system (12) on the domain D.

Remark 8. Note that although the zeros of p_1 given by (13) can be found analytically (since p_1 is a quartic polynomial), it is much more efficient to compute the appropriate zero on $(3c, \tilde{u})$ by some numerical method since the interval of interest provides a good starting point.

The calculation of the optimal parameter and the error:

```
In[∘]:= ξ41o[c_] = (v41o[c] - u41o[c]) / (8 √(1 - c^2));
```

```
η41o[c_] = (v41o[c] + u41o[c])/6;
```

```
In[∘]:= cc = Sqrt[2]/2;
        Print["The pair of optimal parameters is {ξ^,η^}=", {N[ξ41o[cc]], N[η41o[cc]]}, " and the error is ",
         N[ψ41[0, cc, ξ41o[cc], η41o[cc]]], "."]
```

```
The pair of optimal parameters is {ξ^,η^}={0.402742, 1.10845} and the error is 1.93726×10^-6.
```

8. Numerical examples

In this section, several numerical examples will be presented in order to confirm the theoretical results derived in the previous sections. The quality of optimal inner and outer approximants will be compared to the general optimal approximants obtained in [5] where the Hausdorff distance has been considered as a measure of quality. Note that in the case of inner and outer approximants, the radial distance coincides with the Hausdorff one, too.

For the parabolic G^0 case, the optimal parameters and radial distances for several inner angles of a circular arc are collected in Table 1. The optimal parameters and corresponding radial distances for the cubic G^0 and the cubic G^1 case are in Table 2 and Table 3, respectively.

As we have seen in Section 7, the quartic G^1 case is probably the most interesting. Optimal parameters with the corresponding radial distances are in Table 4. There always exist four outer admissible solutions arising from the zeros of p_1 given by (13) but they provide quite different approximants. The best two are on Figure 5. They look quite similar, but the radial errors differ significantly. The other two approximants correspond to two additional sets of parameters. Their graphs are presented on Figure 6. It is seen that they represent the approximation of the complementary circular arc. This is because the radial error at a particular point on the curve is measured in the radial direction from this point to the circle (which is not necessarily a part of the considered circular arc). We guess that these solutions imply the optimal approximants of the complementary arc (see [3]).

Finally, the optimal parameters with the corresponding radial distances for the quartic G^2 case are collected in Table 5.

Error functions and optimal parameters

```
In[∘]:= ψ20[t_, c_, ξ_] = 1/8 (1 - t^2) (2 (1 - t^2) ξ^2 + 4 c (1 + t^2) ξ - 8 + 2 c^2 (1 - t^2));
        ϕ20[t_, c_, ξ_] = Sqrt[ψ20[t, c, ξ] + 1] - 1;
        ξ20i[c_] = 2 - c;
        ξ20o[c_] = 1/c;
        ψ31[t_, c_, ξ_] = 1/16 (1 - t^2)^2 (9 ξ^2 (t^2 c^2 + 1 - c^2) + 12 ξ (2 - t^2) Sqrt[1 - c^2] c - 4 (4 - t^2) (1 - c^2));
        ϕ31[t_, c_, ξ_] = Sqrt[ψ31[t, c, ξ] + 1] - 1;
        ξ31i[c_] = 2/3 Sqrt[1 - c^2] (Sqrt[3 + c^2] - c);
        ξ31o[c_] = (4 Sqrt[1 - c^2])/(3 (1 + c));
        ψ42[t_, c_, ξ_] =
          (t^2 - 1)^3/(64 c^2) (16 (ξ + 1/2 Sqrt[1 - c^2] (Sqrt[3 + c^2] + c))^2 (ξ - 1/2 Sqrt[1 - c^2] (Sqrt[3 + c^2] - c))^2 t^2 -
            16 (ξ - 1/2 (c Sqrt[1 - c^2] - Sqrt[3 + 8 c + 6 c^2 - c^4])) (ξ - 1/2 (c Sqrt[1 - c^2] - Sqrt[3 - 8 c + 6 c^2 - c^4]))
             (ξ - 1/2 (c Sqrt[1 - c^2] + Sqrt[3 - 8 c + 6 c^2 - c^4])) (ξ - 1/2 (c Sqrt[1 - c^2] + Sqrt[3 + 8 c + 6 c^2 - c^4])));
        ϕ42[t_, c_, ξ_] = Sqrt[ψ42[t, c, ξ] + 1] - 1;
        ξ42i[c_] = 1/2 (c Sqrt[1 - c^2] + Sqrt[3 - 8 c + 6 c^2 - c^4]);
```

$$\xi42\mathrm{o}[c\_] = \frac{\sqrt{9-2c^2-3c^4}}{3\sqrt{1-c^2}}\,\mathrm{Cos}\left[\frac{1}{3}\,\mathrm{ArcCos}\left[\frac{2c\left(27-32c^2+9c^4\right)}{\left(-9+2c^2+3c^4\right)\sqrt{9-2c^2-3c^4}}\right]\right]-\frac{c}{3\sqrt{1-c^2}};$$

$$\psi30[t\_, c\_, \xi\_, \eta\_] = \frac{1}{16}\left(t^2-1\right)\left(\left(3\eta-\sqrt{1-c^2}\right)^2 t^4+\left(16\left(1-c^2\right)-9\left(\eta+\sqrt{1-c^2}\right)^2+9\left(\xi-c\right)^2\right)t^2+\left(16-\left(3\xi+c\right)^2\right)\right);$$

$$\phi30[t\_, c\_, \xi\_, \eta\_] = \sqrt{\psi30[t, c, \xi, \eta]+1}-1;$$

$$\xi30\mathrm{i}[c\_] = \frac{32\sqrt{15+c^2}}{81}\,\mathrm{Cos}\left[\frac{1}{3}\,\mathrm{ArcCos}\left[\frac{c\left(63+c^2\right)}{\left(15+c^2\right)^{3/2}}\right]\right]-\frac{47c}{81};$$

$$\eta30\mathrm{i}[c\_] = \frac{27\,\xi30\mathrm{i}[c]^2+10\,c\,\xi30\mathrm{i}[c]+3c^2-40}{8\sqrt{1-c^2}};$$

$$\xi30\mathrm{o}[c\_] = \frac{4-c}{3};$$

$$\eta30\mathrm{o}[c\_] = \frac{3-4c+c^2}{3\sqrt{1-c^2}};$$

$$\psi41[t\_, c\_, \xi\_, \eta\_] = \left(t^2-1\right)^2\left(\frac{1}{64}\left(3\eta-4\sqrt{1-c^2}\,\xi-3c\right)^2 t^4+\frac{1}{64}\left(32c^2\xi^2+32\xi^2-64c\sqrt{1-c^2}\,\xi-18\eta^2+36c\eta-34c^2+16\right)t^2+\frac{1}{64}\left(\left(3\eta+4\sqrt{1-c^2}\,\xi+5c\right)^2-64\right)\right);$$

$$\phi41[t\_, c\_, \xi\_, \eta\_] = \sqrt{\psi41[t, c, \xi, \eta]+1}-1;$$

$$\xi41\mathrm{i}[c\_] = \frac{c\left(1-c^2\right)+\left(1-c\right)^2\sqrt{3+4c+c^2}}{2\sqrt{1-c^2}};$$

$$\eta41\mathrm{i}[c\_] = \frac{1}{3}\left(8-7c+2c^3-2\left(1-c\right)^2\sqrt{3+4c+c^2}\right);$$

$$\mathrm{Clear}[\beta1];$$

$$\beta1[c\_] = -2\left(4c^8+4\left(4+\sqrt{2}\right)c^6-\left(1+6\sqrt{2}\right)c^4+2c^2+3+2\sqrt{2}\right);$$

$$\beta0[u\_, c\_] = 36\left(3-2\sqrt{2}\right)c^9+16\left(3\sqrt{2}-5\right)c^8u+\left(329-198\sqrt{2}+\left(12-8\sqrt{2}\right)u^2\right)c^7+\left(142\sqrt{2}-253\right)c^6u+\left(169-198\sqrt{2}+\left(55-34\sqrt{2}\right)u^2\right)c^5+\left(174\sqrt{2}-51+\left(2\sqrt{2}-3\right)u^2\right)c^4u+\left(4\left(43+9\sqrt{2}\right)-5\left(6\sqrt{2}-7\right)u^2\right)c^3-\left(16\left(7+3\sqrt{2}\right)u+\left(3-2\sqrt{2}\right)u^3\right)c^2+2\left(23+16\sqrt{2}\right)u-4c\left(10+9\sqrt{2}+\sqrt{2}\,u^2\right);$$

$$u41\mathrm{o}[c\_] = \mathrm{If}\left[c == 0,\ 2\sqrt{\frac{1}{7}\left(-1+2\sqrt{2}\right)},\ -\frac{4\sqrt{249+176\sqrt{2}-\left(217+152\sqrt{2}\right)c^2+2c^4}}{3c}\,\mathrm{Cos}\left[\frac{1}{3}\,\mathrm{ArcCos}\left[\frac{\left(5555+3928\sqrt{2}-3\left(2423+1713\sqrt{2}\right)c^2+3\left(478+337\sqrt{2}\right)c^4+2\sqrt{2}\,c^6\right)}{\left(249+176\sqrt{2}-\left(217+152\sqrt{2}\right)c^2+2c^4\right)^{3/2}}\right]-\frac{4\pi}{3}\right]+\frac{-22-16\sqrt{2}+15c^2+4\sqrt{2}\,c^2}{3c}\right];$$

$$\xi41\mathrm{o}[c\_] = \frac{-\beta0[u41\mathrm{o}[c], c]-\beta1[c]*u41\mathrm{o}[c]}{8\,\beta1[c]\sqrt{1-c^2}};$$

$$\eta41\mathrm{o}[c\_] = \frac{-\beta0[u41\mathrm{o}[c], c]+\beta1[c]*u41\mathrm{o}[c]}{6\,\beta1[c]};$$

Optimal quadratic G^0 approximant

```
In[∘]:= angles = {π/2, π/3, π/4, π/6, π/8, π/12};
        Nerror = 3;
        di20 = {};
        do20 = {};
        dd20 = {};
        For[i = 1, i ≤ Length[angles], i++,
         cc = Cos[angles[[i]]];
         AppendTo[di20, {N[ξ20i[cc]], ScientificForm[N[-ϕ20[√2/2, cc, ξ20i[cc]]], Nerror]}];
         If[cc == 0, AppendTo[do20, {"no approximant", "--"}],
          AppendTo[do20, {N[ξ20o[cc]], ScientificForm[N[ϕ20[0, cc, ξ20o[cc]]], Nerror]}]];
         ξξ = ξ /. NSolve[ϕ20[0, cc, ξ] + ϕ20[√(ξ² + cc² - 2)/(ξ - cc), cc, ξ] == 0 && 2 - cc < ξ < cc² - 3 cc + 3, ξ][[1]];
         AppendTo[dd20, {ξξ, ScientificForm[N[ϕ20[0, cc, ξξ]], Nerror]}]
        ]
        Grid[Join[{{φ, "ξ*", "error", "ξ^", "error", "ξ∨", "error"}},
          Table[Flatten[{angles[[i]], dd20[[i]], do20[[i]], di20[[i]]}], {i, Length[angles]}]], Frame → All]
```

Out[∘]=

φ	ξ^*	error	$\xi^\wedge$	error	$\xi^\vee$	error
$\frac{\pi}{2}$	2.21535	1.08×10^{-1}	no approximant	--	2.	1.34×10^{-1}
$\frac{\pi}{3}$	1.54728	2.36×10^{-2}	2.	2.5×10^{-1}	1.5	3.18×10^{-2}
$\frac{\pi}{4}$	1.30843	7.77×10^{-3}	1.41421	6.07×10^{-2}	1.29289	1.08×10^{-2}
$\frac{\pi}{6}$	1.13713	1.58×10^{-3}	1.1547	1.04×10^{-2}	1.13397	2.25×10^{-3}
$\frac{\pi}{8}$	1.07713	5.04×10^{-4}	1.08239	3.14×10^{-3}	1.07612	7.25×10^{-4}
$\frac{\pi}{12}$	1.03427	$1.\times10^{-4}$	1.03528	6.01×10^{-4}	1.03407	1.45×10^{-4}

Optimal cubic G^1 approximant

```
In[∘]:= angles = {π/2, π/3, π/4, π/6, π/8, π/12};
        Nerror = 3;
        di31 = {};
        do31 = {};
        dd31 = {};
        For[i = 1, i ≤ Length[angles], i++,
         cc = Cos[angles[[i]]];
         AppendTo[di31, {N[ξ31i[cc]], ScientificForm[N[-ϕ31[0, cc, ξ31i[cc]]], Nerror]}];
         AppendTo[do31, {N[ξ31o[cc]], ScientificForm[N[ϕ31[1/√3, cc, ξ31o[cc]]], Nerror]}];
         ξξ = ξ /. FindRoot[D[ϕ31[t, cc, ξ], t] == 0 && ϕ31[0, cc, ξ] + ϕ31[t, cc, ξ] == 0, {ξ, ξ31o[cc]}, {t, 0.7}];
         AppendTo[dd31, {ξξ, ScientificForm[-N[ϕ31[0, cc, ξξ]], Nerror]}]
        ]
        Grid[Join[{{φ, "ξ*", "error", "ξ^", "error", "ξ∨", "error"}},
          Table[Flatten[{angles[[i]], dd31[[i]], do31[[i]], di31[[i]]}], {i, Length[angles]}]], Frame → All]
```

Out[∘]=

φ	ξ^*	error	$\xi^\wedge$	error	$\xi^\vee$	error
$\frac{\pi}{2}$	1.31574	1.32×10^{-2}	1.33333	1.84×10^{-2}	1.1547	1.34×10^{-1}
$\frac{\pi}{3}$	0.768087	1.11×10^{-3}	0.7698	1.54×10^{-3}	0.752158	1.15×10^{-2}
$\frac{\pi}{4}$	0.551915	1.96×10^{-4}	0.552285	2.73×10^{-4}	0.548584	1.96×10^{-3}
$\frac{\pi}{6}$	0.35722	1.71×10^{-5}	0.357266	2.39×10^{-5}	0.356822	1.66×10^{-4}
$\frac{\pi}{8}$	0.265206	3.04×10^{-6}	0.265216	4.25×10^{-6}	0.265115	2.92×10^{-5}
$\frac{\pi}{12}$	0.175535	2.67×10^{-7}	0.175537	3.73×10^{-7}	0.175524	2.54×10^{-6}

Optimal quartic G^2 approximant

```
In[•]:= angles = {π/2, π/3, π/4, π/6, π/8, π/12};
Nerror = 3;
di42 = {{N[ξ42i[0]], ScientificForm[N[Limit[-ϕ42[1/2, c, ξ42i[c]], c → 0]], Nerror]}};
do42 = {{N[ξ42o[0]], ScientificForm[N[Limit[ϕ42[0, c, ξ42o[c]], c → 0]], Nerror]}};
dd42 = {{"-", ScientificForm[0.0006952749883544218`, Nerror]}};
For[i = 2, i ≤ Length[angles], i++,
 cc = Cos[angles[[i]]];
 AppendTo[di42, {N[ξ42i[cc]], ScientificForm[N[-ϕ42[1/2, cc, ξ42i[cc]]], Nerror]}];
 ξξ =
  ξ /.
    NSolve[48 Sqrt[1 - cc^2] ξ^3/cc + 48 ξ^2 - 12 Sqrt[1 - cc^2] (3/cc + cc) ξ + 12 (1 - cc^2) == 0 && 1/(6 Sqrt[1 - cc^2]) (-2 cc + Sqrt[9 - 2 cc^2 - 3 cc^4]) < ξ,
      ξ][[1]];
 AppendTo[do42, {N[ξξ], ScientificForm[N[ϕ42[0, cc, ξξ]], Nerror]}];
 ξξ = ξ /. FindRoot[D[ϕ42[t, cc, ξ], t] == 0 && ϕ42[0, cc, ξ] + ϕ42[t, cc, ξ] == 0, {ξ, ξξ}, {t, 0.7}];
 AppendTo[dd42, {ξξ, ScientificForm[N[ϕ42[0, cc, ξξ]], Nerror]}]
]
Grid[Join[{{φ, "ξ*", "error", "ξ^", "error", "ξ∨", "error"}},
  Table[Flatten[{angles[[i]], dd42[[i]], do42[[i]], di42[[i]]}], {i, Length[angles]}]], Frame → All]
```

Out[•]=

φ	ξ^*	error	$\xi^\wedge$	error	$\xi^\vee$	error
$\frac{\pi}{2}$	-	6.95×10^{-4}	0.866025	1.04×10^{-2}	0.866025	9.47×10^{-4}
$\frac{\pi}{3}$	0.547186	2.62×10^{-5}	0.546677	3.62×10^{-4}	0.547225	3.59×10^{-5}
$\frac{\pi}{4}$	0.402587	2.59×10^{-6}	0.402437	3.5×10^{-5}	0.402599	3.56×10^{-6}
$\frac{\pi}{6}$	0.264714	$1.\times10^{-7}$	0.264692	1.33×10^{-6}	0.264716	1.38×10^{-7}
$\frac{\pi}{8}$	0.197577	$1.\times10^{-8}$	0.197572	1.32×10^{-7}	0.197577	1.38×10^{-8}
$\frac{\pi}{12}$	0.131263	3.9×10^{-10}	0.131262	5.1×10^{-9}	0.131263	5.36×10^{-10}

Optimal cubic G^0 approximant

```
In[•]:= angles = {π/2, π/3, π/4, π/6, π/8, π/12};
Nerror = 3;
di30 = {};
do30 = {};
dd30 = {};
For[i = 1, i ≤ Length[angles], i++,
 cc = Cos[angles[[i]]];
 {ξξi, ηηi} = {ξ, η} /. FindRoot[(D[ϕ30[t, cc, ξ, η], t] /. t → 1/2) == 0 && ϕ30[1/2, cc, ξ, η] == 0,
    {ξ, ξ30o[cc]}, {η, η30o[cc]}];
 AppendTo[di30, {ξξi, ηηi, ScientificForm[N[-ϕ30[0, cc, ξξi, ηηi]], Nerror]}];
 AppendTo[do30, {N[ξ30o[cc]], N[η30o[cc]], ScientificForm[N[ϕ30[1/Sqrt[3], cc, ξ30o[cc], η30o[cc]]], Nerror]}];
 {ξξ, ηη} =
  {ξ, η} /. FindRoot[D[ϕ30[t, cc, ξ, η], t] == 0 && ϕ30[0, cc, ξ, η] == ϕ30[t, cc, ξ, η] &&
      D[ϕ30[s, cc, ξ, η], s] == 0 && ϕ30[t, cc, ξ, η] + ϕ30[s, cc, ξ, η] == 0,
     {{ξ, ξ30o[cc]}, {η, η30o[cc]}, {s, 0.6}, {t, 0.9}}];
 AppendTo[dd30, {ξξ, ηη, ScientificForm[-N[ϕ30[0, cc, ξξ, ηη]], Nerror]}]
]
Grid[Join[{{φ, "ξ*", "η*", "error", "ξ^", "η^", "error", "ξ∨", "η∨", "error"}},
```

```
  Table[Flatten[{angles⟦i⟧, dd30⟦i⟧, do30⟦i⟧, di30⟦i⟧}], {i, Length[angles]}]], Frame → All]
```

Out[∘]=

φ	ξ^*	η^*	error	$\xi^\wedge$	$\eta^\wedge$	error	$\xi^\vee$	$\eta^\vee$	error
$\frac{\pi}{2}$	1.32801	0.940495	3.99×10^{-3}	1.33333	1.	1.84×10^{-2}	1.32508	0.925926	6.19×10^{-3}
$\frac{\pi}{3}$	1.16617	0.474943	3.75×10^{-4}	1.16667	0.481125	1.54×10^{-3}	1.16587	0.473285	5.99×10^{-4}
$\frac{\pi}{4}$	1.09754	0.315229	6.84×10^{-5}	1.09763	0.316582	2.73×10^{-4}	1.09748	0.31486	1.1×10^{-4}
$\frac{\pi}{6}$	1.04465	0.190431	6.11×10^{-6}	1.04466	0.190599	2.39×10^{-5}	1.04465	0.190384	9.89×10^{-6}
$\frac{\pi}{8}$	1.02537	0.137616	1.09×10^{-6}	1.02537	0.137655	4.25×10^{-6}	1.02537	0.137605	1.77×10^{-6}
$\frac{\pi}{12}$	1.01136	0.0892586	9.65×10^{-8}	1.01136	0.0892636	3.73×10^{-7}	1.01136	0.0892572	1.57×10^{-7}

Optimal quartic G^1 approximant

In[∘]:=

```
angles = {π/2, π/3, π/4, π/6, π/8, π/12};
Nerror = 3;
di41 = {};
do41 = {};
dd41 = {};
For[i = 1, i ≤ Length[angles], i++,
 cc = N[Cos[angles⟦i⟧]];
 {ξξ, ηη} =
  {ξ, η} /. FindRoot[D[ϕ41[t, cc, ξ, η], t] == 0 && ϕ41[0, cc, ξ, η] == ϕ41[t, cc, ξ, η] &&
      D[ϕ41[s, cc, ξ, η], s] == 0 && ϕ41[t, cc, ξ, η] + ϕ41[s, cc, ξ, η] == 0,
     {{ξ, ξ41o[cc]}, {η, η41o[cc]}, {s, 0.4}, {t, 0.8}}];
 AppendTo[di41, {ξ41i[cc], η41i[cc], ScientificForm[N[-ϕ41[1/2, cc, ξ41i[cc], η41i[cc]]], Nerror]}];
 AppendTo[do41, {ξ41o[cc], η41o[cc], ScientificForm[N[ϕ41[0, cc, ξ41o[cc], η41o[cc]]], Nerror]}];
 AppendTo[dd41, {ξξ, ηη, ScientificForm[N[ϕ41[0, cc, ξξ, ηη]], Nerror]}]
]
Grid[Join[{{φ, "ξ*", "η*", "error", "ξ^", "η^", "error", "ξ∨", "η∨", "error"}},
  Table[Flatten[{angles⟦i⟧, dd41⟦i⟧, do41⟦i⟧, di41⟦i⟧}], {i, Length[angles]}]], Frame → All]
```

Out[∘]=

φ	ξ^*	η^*	error	$\xi^\wedge$	$\eta^\wedge$	error	$\xi^\vee$	$\eta^\vee$	error
$\frac{\pi}{2}$	0.871525	1.50505	1.57×10^{-4}	0.87247	1.50401	2.4×10^{-4}	0.866025	1.51197	9.47×10^{-4}
$\frac{\pi}{3}$	0.547788	1.20082	6.21×10^{-6}	0.547886	1.20071	9.59×10^{-6}	0.547225	1.20145	3.59×10^{-5}
$\frac{\pi}{4}$	0.402721	1.10847	6.25×10^{-7}	0.402742	1.10845	9.69×10^{-7}	0.402599	1.10858	3.56×10^{-6}
$\frac{\pi}{6}$	0.264731	1.0468	2.45×10^{-8}	0.264734	1.0468	3.8×10^{-8}	0.264716	1.04681	1.38×10^{-7}
$\frac{\pi}{8}$	0.197581	1.02605	2.46×10^{-9}	0.197582	1.02605	3.82×10^{-9}	0.197577	1.02605	1.38×10^{-8}
$\frac{\pi}{12}$	0.131263	1.01149	9.6×10^{-11}	0.131264	1.01149	1.49×10^{-10}	0.131263	1.01149	5.36×10^{-10}