THE HOMOTOPY TYPE OF $BG_2^\wedge$ FOR SOME SMALL MATRIX GROUPS G

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Abstract. Let q be a power of an odd prime. We prove that the mod-2 cohomologies of $BGL_2(\mathbb{F}_q)_2^\wedge$, $BSL_3(\mathbb{F}_q)_2^\wedge$, and $BGL_3(\mathbb{F}_q)_2^\wedge$, as algebras over the mod-2 Steenrod algebra, together with the associated Bockstein spectral sequence, determine the homotopy types of respectively $BGL_2(\mathbb{F}_q)_2^\wedge$, $BSL_3(\mathbb{F}_q)_2^\wedge$, and $BGL_3(\mathbb{F}_q)_2^\wedge$.

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1. Introduction

Let G and H be finite groups which have the same mod-p cohomology as algebras over the mod-p Steenrod algebra $\mathcal{A}_p$. The question whether the p-completions $BG_p^\wedge$ and $BH_p^\wedge$ are homotopy equivalent, has a negative answer in general. For example, all cyclic groups $\mathbb{Z}/p^n$ for $n \geq 2$ have the same mod-p cohomology but their classifying spaces $B\mathbb{Z}/p^n$, the lens spaces $L_{p^n}^\infty$, are not homotopy equivalent. The cohomology of the group $\mathbb{Z}/p$ is different from that of the group $\mathbb{Z}/p^n$ for $n \geq 2$, since in the case $\mathbb{Z}/p$ the Bockstein homomorphism maps the generator of cohomology in dimension $2k-1$ to the generator in dimension $2k$ for all $k \in \mathbb{N}$. The homotopy type of the space $B\mathbb{Z}/p$ is determined up to p-completion by $H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ considered as an algebra over $\mathcal{A}_p$. In the case $\mathbb{Z}/p^n$, $n \geq 2$, the higher Bockstein operator β_n connects generators in dimensions $2k-1$ and $2k$. One might thus wonder if mod-p cohomology of a finite group G as an algebra over $\mathcal{A}_p$, together with the higher Bockstein operators, determines the homotopy type of $BG_p^\wedge$. So the cohomology of a space is considered as an object in the category $\mathcal{K}_\beta$ of unstable algebras over $\mathcal{A}_p$ together with higher Bockstein homomorphisms (see section 2). We say that spaces X and Y are comparable if $H^*(X;\mathbb{F}_p)$ and $H^*(Y;\mathbb{F}_p)$ are isomorphic objects in $\mathcal{K}_\beta$. We say that the homotopy type of a p-complete space X is determined by its mod-p cohomology if any p-complete space Y, comparable to X, is homotopy equivalent to X. There are some finite groups G for which the p-completions of their classifying space $BG_p^\wedge$, are determined by their mod-p cohomology: finite abelian groups, $SL_2(\mathbb{F}_q)$ and $PSL_2(F_q)$ at prime $p=2$ for an odd prime power q (see [6]), the dihedral groups D_{2^n}, the extra special groups ([7]), and the generalized quaternion groups Q_{2^n} ([7], [8]). In this paper we prove the following theorem.

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Theorem 1.1. *Let q be a power of an odd prime. The spaces $BGL_2(\mathbb{F}_q)^\wedge_2$, $BSL_3(\mathbb{F}_q)^\wedge_2$, and $BGL_3(\mathbb{F}_q)^\wedge_2$ are determined by their mod-2 cohomology.*

2. Conventions and Terminology

All spaces considered are assumed to have the homotopy type of a CW complex. For a given space X we write $H^*(X)$ for its mod-2 cohomology $H^*(X;\mathbb{F}_2)$, and $X^\wedge_2$ denotes $\mathbb{F}_{2^\infty}$-completion or 2-completion of the space X in the sense of Bousfield and Kan [4]. As in the previous section $\mathcal{A}_2$ denotes the mod-2 Steenrod algebra, and $\mathcal{K}_2$ denotes the category of unstable algebras over $\mathcal{A}_2$. A Bockstein spectral sequence attached to an arbitrary unstable algebra is not widely used, hence we will recall the definition.

Definition 2.1. [7] Let K be an unstable algebra over $\mathcal{A}_2$. A Bockstein spectral sequence for K is a spectral sequence $\{E_n(K),\beta_n\}_{n=1}^\infty$ of differential graded algebras, where the differentials have degree one, and such that

(1) $E_1(K)=K$ and $\beta_1=Sq^1$ is the primary Bockstein operator,
(2) if $x\in K^{even}$ and $x^2\neq 0$ in $E_2(K)$, then $\beta_2(x^2)=xSq^1x+Sq^{|x|}Sq^1x$,
(3) if $x\in E_n(K)^{even}$ and $x^2\neq 0$ in $E_{n+1}(K)$, $n\geq 2$ then $\beta_{n+1}(x^2)=x\beta_n(x)$.

Let $\mathcal{K}_\beta$ be the category whose objects are pairs $K_\beta=(K;\{E_n(K),\beta_n\}_{n=1}^\infty)$, where K is an unstable algebra over $\mathcal{A}_2$ and $\{E_n(K),\beta_n\}_{n=1}^\infty$ an associated Bockstein spectral sequence. A morphism $f\colon K_\beta\longrightarrow K'_\beta$ in $\mathcal{K}_\beta$ is a family of morphisms $\{f_n\}_{n=1}^\infty$, where $f_1\colon K\longrightarrow K'$ is a morphism in $\mathcal{K}_2$, and for each $n\geq 2$, $f_n\colon E_n(K)\longrightarrow E_n(K')$ is a morphism of differential graded algebras, which is induced by f_{n-1}. The mod-2 cohomology of a space X together with its natural Bockstein spectral sequence as an object in $\mathcal{K}_\beta$ will be denoted by $H^*_\beta(X)$.

3. The Homotopy type of $BGL_2(\mathbb{F}_q)^\wedge_2$

In this section we will prove that $BGL_2(\mathbb{F}_q)^\wedge_2$ is determined by its mod-2 cohomology. The group $GL_2(\mathbb{F}_q)$ has order $q(q-1)^2(q+1)$ and the mod-2 cohomology of $BGL_2(\mathbb{F}_q)$ depends on q. If $q\equiv 1 \pmod 4$, then

$$H^*(BGL_2(\mathbb{F}_q))=\mathbb{F}_2[a_2,a_4]\otimes E(b_1,b_3) \tag{1}$$

and the action of the Steenrod algebra is defined as follows:

	b_1	a_2	b_3	a_4
Sq^1	0	0	0	0
Sq^2	0	a_2^2	$a_2b_3+b_1a_4$	a_2a_4

and $\beta_s(b_1)=a_2$, where $2^s\|(q-1)$ (the symbol $2^s\|n$ means that 2^s is the highest power of 2 dividing n) ([13, IV Theorem 8.1], [17, Theorem 1.3]). By [17, Theorem 2.3], $H^4(BGL_2(\mathbb{F}_q);\mathbb{Z}[\frac{1}{p}])=\mathbb{Z}/(q^2-1)\times\mathbb{Z}/(q-1)$, where q is a power of the prime p. Hence $\beta_{s+1}(b_3)=a_4$.

If $q\equiv 3 \pmod 4$, then

$$H^*(BGL_2(\mathbb{F}_q))=\mathbb{F}_2[b_1,b_3,a_4]/(b_1^6+b_3^2+a_4b_1^2) \tag{2}$$

and the action of the Steenrod algebra is defined as follows:

	b_1	b_3	a_4
Sq^1	b_1^2	b_1^4	0
Sq^2	0	$b_1^2b_3 + b_1a_4$	$b_1^2a_4$

([13, Theorem 8.2], [17, Theorem 1.3]) and $\beta_{s+1}(b_1^3 + b_3) = a_4$ where $2^s \| (q+1)$ [17, Theorem 2.3].

Let q be any odd prime power and let X be a 2-complete space such that $H^*_\beta(X) \cong H^*_\beta(BGL_2(\mathbb{F}_q))$. Let $2^s \| (q-1)$ and let $g\colon X \longrightarrow B\mathbb{Z}/2^s$ be a map such that g^* maps the generator of $H^1(B\mathbb{Z}/2^s)$ to the generator of $H^1(X)$. Let Y be the homotopy fiber of the map g. Using the Eilenberg-Moore spectral sequence we see that $H^*_\beta(Y) \cong H^*_\beta(BSL_2(\mathbb{F}_q))$. Because Y is 2-complete and $BSL_2(\mathbb{F}_q)^\wedge_2$ is determined by its mod-2 cohomology [6], Y is homotopy equivalent to $BSL_2(\mathbb{F}_q)^\wedge_2$.

Homotopy classes of fibrations with base space $B\mathbb{Z}/2^s$ and fiber $BSL_2(\mathbb{F}_q)^\wedge_2$ are in bijection with group extensions of the form $SL_2(\mathbb{F}_{3^{2^t}}) \longrightarrow \cdot \longrightarrow \mathbb{Z}/2^s$ for some t such that $SL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_{3^{2^t}})$ have Sylow 2-subgroups of the same order if $q \equiv \pm 1 \pmod 8$ and $t = 0$ otherwise [8, Corollary 6.5].

The group $\mathrm{Out}(SL_2(\mathbb{F}_{3^{2^t}}))$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2^t$ for $t \geq 1$ and to $\mathbb{Z}/2$ for $t = 0$ [14, Theorem 2.5.12]. The generator of the factor $\mathbb{Z}/2$ corresponds to conjugation by a matrix in $GL_2(\mathbb{F}_{3^{2^t}})$ and the elements in $\mathbb{Z}/2^t$ correspond to the Frobenius homomorphisms; i.e. the generator of $\mathbb{Z}/2^t$ maps a matrix A to the matrix where all entries of A are replaced by their cubes. The group $\mathbb{Z}/2^s$ acts on the center $Z(SL_2(\mathbb{F}_{3^{2^t}})) = \mathbb{Z}/2$ trivially. Because $H^2(\mathbb{Z}/2^s; Z(SL_2(\mathbb{F}_{3^{2^t}}))) = \mathbb{Z}/2$, for each action $\psi\colon \mathbb{Z}/2^s \longrightarrow \mathrm{Out}(SL_2(\mathbb{F}_q))$ there are two extensions of group $\mathbb{Z}/2^s$ by $SL_2(\mathbb{F}_{3^{2^t}})$, inducing the action ψ [5, Theorem 6.6]. The two extensions H_ψ and K_ψ have the same elements as $SL_2(\mathbb{F}_{3^{2^t}}) \times \mathbb{Z}/2^s$ and the operations are defined as

$$(A, \zeta^a)(B, \zeta^b) := (A\hat{\psi}(B), \zeta^{a+b}),$$
$$(A, \zeta^a)(B, \zeta^b) := (A\hat{\psi}(B)f(\zeta^a, \zeta^b), \zeta^{a+b}),$$

where $\hat{\psi} \in \mathrm{Aut}(SL_2(\mathbb{F}_{3^{2^t}}))$ is any representative of ψ, ζ is a generator of the group $\mathbb{Z}/2^s < \mathbb{F}_q^*$, and $f\colon \mathbb{Z}/2^s \times \mathbb{Z}/2^s \longrightarrow Z(SL_2(\mathbb{F}_q)) = \{I, -I\}$ is a factor set defined as

$$f(\zeta^a, \zeta^b) = \begin{cases} I & ; a+b \pmod{2^{s+1}} < 2^s, \\ -I & ; a+b \pmod{2^{s+1}} \geq 2^s. \end{cases}$$

We will show that only one extension has the mod-2 cohomology isomorphic to the mod-2 cohomology of the group $GL_2(\mathbb{F}_{3^{2^t}})$. This shows that $BGL(\mathbb{F}_{3^{2^t}})^\wedge_2$ is determined by its mod-2 cohomology.

Let $SL_2(\mathbb{F}_{3^{2^t}}) \longrightarrow L \longrightarrow \mathbb{Z}/2^s$ be an extension that induces an action ψ, which is neither the trivial action nor conjugation by an element in $GL_2(\mathbb{F}_{3^{2^t}})$. This implies that $t \geq 1$, because for $t = 0$ the group $\mathrm{Out}(SL_2(\mathbb{F}_3)) = \mathbb{Z}/2$. Since $2^{t+2} \| 3^{2^t} - 1$, $s = t+2$. Because $H^q(BSL_2(\mathbb{F}_{3^{2^t}}); \mathbb{Z}[\frac{1}{3}]) = 0$ for $q = 1, 2, 3$ [17, Theorem 2.3], the elements $E_2^{p,q}$ of the Serre spectral sequence of the fibration $BSL_2(\mathbb{F}_{3^{2^t}}) \longrightarrow BL \longrightarrow B\mathbb{Z}/2^s$

vanish for $q=1,2,3$. And also $E_2^{5,0}=H^5(B\mathbb{Z}/2^s;H^0(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\frac{1}{3}]))=0$, hence

$$\begin{aligned}H^4(BL;\mathbb{Z}[\tfrac{1}{3}]) &= E_2^{4,0}\oplus E_2^{0,4}=\\ &= H^4(\mathbb{Z}/2^s;H^0(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\tfrac{1}{3}]))\oplus H^0(\mathbb{Z}/2^s;H^4(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\tfrac{1}{3}]))=\\ &= \mathbb{Z}/2^s\oplus H^4(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\tfrac{1}{3}])^{\mathbb{Z}/2^s},\end{aligned}$$

where $H^4(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\frac{1}{3}])^{\mathbb{Z}/2^s}$ is the fixed-point set of the action induced by ψ. Let $x\in H^4(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\frac{1}{3}])=\mathbb{Z}/2^{t+3}$ [17, Theorem 2.3] be a generator. Let $i\colon \mathbb{Z}/2^{t+2}\longrightarrow SL_2(\mathbb{F}_{3^{2^t}})$ be inclusion defined as $i(\zeta^k)=\mathrm{Diag}(\zeta^k,\zeta^{-k})$. Because i induces an isomorphism from $H^4(BSL_2(\mathbb{F}_{3^{2^t}}))$ to $H^4(B\mathbb{Z}/2^{t+2})$, the element $i^*(x)$ is a generator of $H^4(B\mathbb{Z}/2^{t+2};\mathbb{Z}[\frac{1}{3}])=\mathbb{Z}/2^{t+2}$. The restriction of the action ψ on the subgroup $\mathbb{Z}/2^{t+2}$ is powering by 3^{2^r} for some $r\in\{1,\dots,t-1\}$. Then $i^*(x)$ is not fixed by this action, therefore $H^4(BSL_2(\mathbb{F}_{3^{2^t}});\mathbb{Z}[\frac{1}{3}])^{\mathbb{Z}/2^s}\neq\mathbb{Z}/2^{t+3}$. We see that the mod-2 cohomology of $BL_2^\wedge$ differs from the mod-2 cohomology of $BGL_2(\mathbb{F}_{3^{2^t}})$.

Let ψ be the trivial action or conjugation by an element in $GL_2(\mathbb{F}_{3^{2^t}})$. The maximal elementary 2-subgroup of K_ψ has rank 1, and because the maximal elementary 2-subgroup of $GL_2(\mathbb{F}_{3^{2^t}})$ has rank 2, the mod-2 cohomology of BK_ψ differs from the mod-2 cohomology of $BGL_2(\mathbb{F}_q)$ [12]. Also if ψ is trivial, the mod-2 cohomology of $H_\psi=\mathrm{SL}_2(\mathbb{F}_q)\times\mathbb{Z}/2^s$ differs from the mod-2 cohomology of $BGL_2(\mathbb{F}_q)$. Therefore X is homotopy equivalent to $BGL_2(\mathbb{F}_q)^\wedge_2$.

4. The Homotopy type of $BSL_3(\mathbb{F}_q)^\wedge_2$

The group $SL_3(\mathbb{F}_q)$ has order $q^3(q-1)^2(q^2+q+1)(q+1)$. If $q\equiv 3 \pmod 4$, the mod-2 cohomology of $BSL_3(\mathbb{F}_q)$ is

$$H^*(BSL_3(\mathbb{F}_q))=\mathbb{F}_2[v_3,v_4,v_5]/(v_3^2v_4+v_5^2), \tag{3}$$

and the action of the Steenrod algebra is defined as follows:

	v_3	v_4	v_5
Sq^1	0	0	v_3^2
Sq^2	v_5	v_3^2	0
Sq^4	0	v_4^2	$v_3^3+v_4v_5$

and $\beta_{s+1}(v_3)=v_4$, where $2^s||(q+1)$ ([13, IV, Theorem 8.2] and [17, Theorem 1.3, Theorem 2.3]). If $q\equiv 1 \pmod 4$ then

$$H^*(BSL_3(\mathbb{F}_q))=\mathbb{F}_2[v_4,v_6]\otimes E(v_3,v_5), \tag{4}$$

and the action of the Steenrod algebra is defined as follows:

	v_3	v_4	v_5	v_6
Sq^1	0	0	0	0
Sq^2	v_5	v_6	0	0
Sq^4	0	v_4^2	$v_3v_6+v_4v_5$	v_4v_6

,

$\beta_{s+1}(v_3)=v_4$ and $\beta_s(v_5)=v_6$, where $2^s||(q-1)$ ([13, IV, Theorem 8.1] and [17, Theorem 1.3, Theorem 2.3]).

To prove homotopy uniqueness of $BSL_3(\mathbb{F}_q)_2^\wedge$ we will use its centralizer homology decomposition. Let $\mathcal{A}_2(SL_3(\mathbb{F}_q))^{op}$ be the Quillen category of the group $BSL_3(\mathbb{F}_q)$. This is the category with objects nontrivial elementary abelian 2-subgroups of $SL_3(\mathbb{F}_q)$, and a morphism $c_g\colon E_1 \longrightarrow E_2$ is a homomorphism which is the restriction of an inner automorphism of $SL_3(\mathbb{F}_q)$; i.e. $c_g(x) = gxg^{-1}$ for some $q \in SL_3(\mathbb{F}_q)$. Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}_2(SL_3(\mathbb{F}_q))^{op}$. The centralizer diagram

$$\alpha\colon \mathcal{C} \longrightarrow \mathit{Spaces}$$

is the functor which sends every object U to a model of the classifying space

$$ESL_3(\mathbb{F}_q) \times_{SL_3(\mathbb{F}_q)} (SL_3(\mathbb{F}_q)/C_{SL_3(\mathbb{F}_q)}(U)) \simeq BC_{SL_3(\mathbb{F}_q)}(U)$$

of its centralizer. We say that $\mathcal{C}$ is an ample collection if the natural map

$$\operatorname*{hocolim}_{C} \alpha \longrightarrow BSL_3(\mathbb{F}_q)$$

is a mod-2 homology isomorphism.

Let $A = \mathrm{Diag}(-1,-1,1)$ and $B = \mathrm{Diag}(-1,1,-1)$ be diagonal matrices in $SL_3(\mathbb{F}_q)$. Consider the following elementary abelian 2-subgroups of $SL_3(\mathbb{F}_q)$, generated by A, and by A and B: $V = \langle A\rangle, W = \langle A, B\rangle$. Let $\mathbb{A}$ be the full subcategory of the Quillen category $\mathcal{A}_2(SL_3(\mathbb{F}_q))^{op}$ which has objects $\mathcal{E} = \{V, W\}$. Because every elementary abelian 2-subgroup of $SL_3(\mathbb{F}_q)$ is isomorphic to one of the elements in $\mathcal{E}$, the category $\mathbb{A}$ is an ample collection of elementary abelian 2-subgroups of $SL_3(\mathbb{F}_q)$ [16, Theorem 7.7].

The centralizers of the objects in $\mathcal{E}$ are $C_{SL_3(\mathbb{F}_q)}(V) = GL_2(\mathbb{F}_q)$ and $C_{SL_3(\mathbb{F}_q)}(W) = (\mathbb{Z}/(q-1))^2$ (the subgroup of all diagonal matrices). The normalizers are $N_{SL_3(\mathbb{F}_q)}(V) = GL_2(\mathbb{F}_q)$ and $N_{SL_3(\mathbb{F}_q)}(W) = (\mathbb{Z}/(q-1))^2 \rtimes \Sigma_3$, where the action of the permutation group Σ_3 on $(\mathbb{Z}/(q-1))^2$ is defined as follows: we look at the group $(\mathbb{Z}/(q-1))^2$ as a subgroup of $(\mathbb{Z}/(q-1))^3$ of those triples (t_1, t_2, t_3) for which $t_1 + t_2 + t_3 \equiv 0 \pmod{(q-1)}$, and the action of the group Σ_3 on $(\mathbb{Z}/(q-1))^3$ by permutation induces the action of Σ_3 on $(\mathbb{Z}/(q-1))^2$. So the morphisms in $\mathbb{A}$ are $\mathrm{Mor}(V, V) = N_{SL_3(\mathbb{F}_q)}(V)/C_{SL_3(\mathbb{F}_q)}(V) = 1$, $\mathrm{Mor}(W, W) = N_{SL_3(\mathbb{F}_q)}(W)/C_{SL_3(\mathbb{F}_q)}(W) = \Sigma_3$, and $\mathrm{Mor}(V, W) = N_{SL_3(\mathbb{F}_q)}(V, W)/C_{SL_3(\mathbb{F}_q)}(V) = \Sigma_3/\Sigma_2$. We can picture the category $\mathbb{A}$ as

$$V \underset{\Sigma_3/\Sigma_2}{\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}}} W \circlearrowleft \Sigma_3.$$

The 2-completion of the diagram $\alpha\colon \mathbb{A} \longrightarrow \mathit{Spaces}$ is

$$BGL_2(\mathbb{F}_q)_2^\wedge \underset{\Sigma_3/\Sigma_2}{\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}} (B(\mathbb{Z}/q-1)^2)_2^\wedge \circlearrowleft \Sigma_3.$$

By [16, Theorem 7.7], the natural map

$$\operatorname*{hocolim}_{\mathbb{A}} \alpha_{\mathcal{E}} \longrightarrow BSL_3(\mathbb{F}_q)$$

is a mod-2 cohomology isomorphism, hence $(\operatorname{hocolim}_{\mathbb{A}} \alpha_{\mathcal{E}})_2^\wedge \cong BSL_3(\mathbb{F}_q)_2^\wedge$.

Now we will prove that $BSL_3(\mathbb{F}_q)^\wedge_2$ is determined by its mod-2 cohomology. Let X be a 2-complete space such that $H^*_\beta(X) \cong H^*_\beta(BSL_3(\mathbb{F}_q))$. From the above discussion we see that to construct a map $BSL_3(\mathbb{F}_q)^\wedge_2 \longrightarrow X$ it is enough to define a family of maps $BC_{SL_3(\mathbb{F}_q)}(U) \longrightarrow X$, $U \in \mathcal{E}$, which with some compatibility assumption will define a map $(\operatorname{hocolim}_{\mathbb{A}} \alpha_{\mathcal{E}}) \longrightarrow X$. Hence we need to define two maps $f_V\colon BGL_2(\mathbb{F}_q) \longrightarrow X$ and $f_W\colon BW \longrightarrow X$. By Lannes' theory [19], there is a map $f'_W\colon BW \longrightarrow X$ such that $(f'_W)^*$ equals the composite $H^*(X) \cong H^*(BSL_3(\mathbb{F}_q)) \xrightarrow{Bi^*_W} H^*(BW)$. Define f_W as the composite $B(\mathbb{Z}/(q-1))^2 \longrightarrow (B(\mathbb{Z}/(q-1))^2)^\wedge_2 = BW \longrightarrow X$. For $U = V$ we use the following proposition.

Proposition 4.1. *Let X be a 2-complete space and $H^*_\beta(X) \cong H^*_\beta(BSL_3(\mathbb{F}_q))$. Then there exists a map $\bar{f}_V\colon BV \longrightarrow X$, such that* $\operatorname{Map}(BV, X)_{\bar{f}_V}$ *is homotopy equivalent to $BGL_2(\mathbb{F}_q)^\wedge_2$.*

Proof. By Lannes' theory [19], there exists a map $\bar{f}_V\colon BV \longrightarrow X$ such that $\bar{f}^*_V$ equals the composite $H^*(X) \cong H^*(BSL_3(\mathbb{F}_q)) \xrightarrow{Bi^*_V} H^*(BV)$. We will prove that the cohomology of $\operatorname{Map}(BV, X)_{\bar{f}_V}$ is isomorphic to $H^*_\beta(BGL_2(\mathbb{F}_q))$ as an object in $\mathcal{K}_\beta$.

By [19, Proposition 3.4.6.],

$$T^V_{Bi^*_V} H^*(BSL_3(\mathbb{F}_q)) \cong H^*(BC_{SL_3(\mathbb{F}_q)}(V)) = H^*(BGL_2(\mathbb{F}_q)),$$

where $T^V_{Bi^*_V}$ is the Lannes' functor. If $q \equiv 3 \pmod 4$ then

$$T^V_{\bar{f}^*_V} H^*(X) \cong T^V_{Bi^*_V} H^*(BSL_3(\mathbb{F}_q)) \cong H^*(BGL_2(\mathbb{F}_q))$$

is free in degrees ≤ 2, which means that the map

$$(H^1(BGL_2(\mathbb{F}_q)) \otimes H^1(BGL_2(\mathbb{F}_q)))_{\Sigma_2} \longrightarrow H^2(BGL_2(\mathbb{F}_q))$$

induced by the product on $H^*(BGL_2(\mathbb{F}_q))$ is injective, hence, by [19, Théorème 3.2.4],

$$H^*(\operatorname{Map}(BV, X)_{\bar{f}_V}) \cong T^V_{\bar{f}^*_V} H^*(X) \cong H^*(BGL_2(\mathbb{F}_q))$$

and the evaluation map $e\colon \operatorname{Map}(BV, X)_{\bar{f}_V} \longrightarrow X$ induces the map on the mod-2 cohomology which equals the composite $H^*(X) \cong H^*(BSL_3(\mathbb{F}_q)) \xrightarrow{Bi^*_{GL_2(\mathbb{F}_q)}} H^*(BGL_2(\mathbb{F}_q))$.

If $q \equiv 1 \pmod 4$, then $T^V_{Bi^*_V} H^*(BSL_3(\mathbb{F}_q)) \cong H^*(BC_{SL_3(\mathbb{F}_q)}(V)) = H^*(BGL_2(\mathbb{F}_q))$ is not free in degrees ≤ 2, hence Lannes' theory does not guarantee that $T^V_{\bar{f}^*_V} H^*(X)$ is isomorphic $H^*(\operatorname{Map}(BV, X)_{\bar{f}_V})$. By [1, Theorem 3], we can use the Lannes' T functor if Y is of finite type such that $H^1(Y) = 0$ and $\bar{f}_V\colon BV \longrightarrow Y$ is finitely T-representable; i.e. there exists an increasing sequence $\alpha(s)$ and a map of towers $g\colon \{\operatorname{Map}(BV, P_{\alpha(s)} Y^\wedge_2)_{f_s}\} \longrightarrow \{K(G_s, 1)\}$, where

(1) $P_{\alpha(s)}Y$ is the $\alpha(s)^{th}$ Postnikov stage and f_s the map induced by $\bar{f}_V$,
(2) $T^V_{\bar{f}^*_V} H^*(Y)$ is of finite type,
(3) G_s a finite 2-group for all s,
(4) $G_\infty = \varprojlim G_s$ is a finite 2-group or $H^*(G_\infty)$ is of finite type and $\operatorname{Tor}^{*,*}_{H^*(G_\infty)}(T^V_{\bar{f}^*_V} H^*(Y))$ is finite-dimensional in each total degree,

(5) the map g induces a pro-isomorhism in H_1 and a pro-epimorphism in H_2, and
(6) $H^*(\varprojlim G_s) \cong \varinjlim H^*(G_s)$, induced by the natural map.

We will show that $\bar{f}_V$ is finitely T-representable.

An n-approximation for a connected algebra A over the Steenrod algebra is a sequence $C \longrightarrow B \longrightarrow A$ of connected algebras over the Steenrod algebra for which the composite is trivial in positive degrees and the induced map $B//C \longrightarrow A$ is a bijection in degrees less then n and an injection in degrees bigger than or equal to n. The sequence

$$\mathbb{F}_2[a_2] \longrightarrow \mathbb{F}_2[b_1] \longrightarrow T^V_{\bar{f}^*_V} H^*(X)$$

is a 2-approximation of $T^V_{\bar{f}^*_V} H^*(X) = H^*(BG_2(\mathbb{F}_q)) = \mathbb{F}_2[a_2, a_4] \otimes E(b_1, b_3)$. If this sequence were actually a 3-approximation of $T^V_{\bar{f}^*_V} H^*(X)$, then $\bar{f}_V$ would be finitely T-representable [1, Theorem 6], but this is not the case here. But $\mathbb{F}_2[a_2]//\mathbb{F}_2[b_1] \cong E(b_1)$ is an exterior algebra with one generator in dimension 1, so by [1, Example 12 and Theorem 16], $\bar{f}_V$ is finitely T-representable. Hence by [1, Theorem 3],

$$H^*(\mathrm{Map}(BV, X)_{\bar{f}_V}) \cong T^V_{\bar{f}^*_V} H^*(X) \cong H^*(BGL_2(\mathbb{F}_q))$$

and the evaluation map $e\colon \mathrm{Map}(BV, X)_{\bar{f}_V} \longrightarrow X$ induces the map which equals the composite $H^*(X) \cong H^*(BSL_3(\mathbb{F}_q)) \xrightarrow{Bi^*_{GL_2(\mathbb{F}_q)}} H^*(BGL_2(\mathbb{F}_q))$.

To finish the proof, we have to show that $H^*_\beta(\mathrm{Map}(BV, X)_{\bar{f}_V})$ and $H^*_\beta(BGL_2(\mathbb{F}_q))$ are isomorphic as objects in $\mathcal{K}_\beta$.

Let $q \equiv 3 \pmod 4$. In the diagram

$$\begin{array}{ccccc}
GL_2(\mathbb{F}_q) & \xrightarrow{\bar{e}} & SL_3(\mathbb{F}_q) & \xrightarrow{i} & GL_3(\mathbb{F}_q) \\
\uparrow{\scriptstyle i_2} & & & & \uparrow{\scriptstyle i_3} \\
(\mathbb{Z}/2)^2 & = & (\mathbb{Z}/2)^2 & \xrightarrow{j} & (\mathbb{Z}/2)^3
\end{array}$$

both vertical arrows are maps to diagonal matrices and $j(t_1, t_2) = (t_1, t_2, t_1t_2)$. By [13, IV Theorem 8.2], the map

$$Bi_2^*\colon \mathbb{F}_2[b_1, b_3, a_4]/(b_1^6 + b_3^2 + a_4 b_1^2) \longrightarrow \mathbb{F}_2[x_1, x_2]$$

is defined by $Bi_2^*(b_1) = x_1 + x_2$, $Bi_2^*(b_3) = x_1^3 + x_2^3$, and $Bi_2^*(a_4) = x_1^2 x_2^2$ and the map

$$Bi_3^*\colon \mathbb{F}_2[v_1, v_3, v_4, v_5]/(v_1^4 v_3^2 + v_1^6 v_4 + v_3^2 v_4 + v_5^2) \longrightarrow \mathbb{F}_2[y_1, y_2, y_3]$$

is defined by $Bi_3^*(v_1) = y_1 + y_2 + y_3$, $Bi_3^*(v_3) = y_1^3 + y_2^3 + y_3^3$, $Bi_3^*(v_4) = y_1^2 y_2^2 + y_1^2 y_3^2 + y_2^2 y_3^2$ and $Bi_3^*(v_5) = y_1^5 + y_2^5 + y_3^5$. Because the map Bi^* is surjective [17, Theorem 1.3] and the map $Bj^*\colon \mathbb{F}_2[y_1, y_2, y_3] \longrightarrow \mathbb{F}_2[x_1, x_2]$ is defined by $Bj^*(y_1) = x_1$, $Bj^*(y_2) = x_2$ and $Bj^*(y_3) = x_1 + x_2$, the map $e^* = B\bar{e}^*$ is defined by $e^*(v_3) = b_3 + b_1^3$, $e^*(v_4) = a_4 + b_1^4$, and $e^*(v_5) = b_1^2 b_3 + b_1 a_4 + b_1^5$. Hence at the $(s+1)^{st}$ stage of the Bockstein spectral sequence, we get

$$a_4 = e^*(v_4) = e^*(\beta_{s+1}(v_3)) = \beta_{s+1}(e^*(v_3)) = \beta_{s+2}(b_3 + b_1^3).$$

Therefore $H^*_\beta(\mathrm{Map}(BV, X)_{\bar{f}_V})$ is isomorphic $H^*_\beta(BGL_2(\mathbb{F}_q))$.

If $q \equiv 1 \pmod 4$, then in a similar way as above we calculate that the map e^* is defined by $e^*(v_3) = b_3$, $e^*(v_4) = a_2^2 + a_4$, $e^*(v_5) = b_1a_4 + b_3a_2$, and $e^*(v_6) = a_2a_4$. Hence, at the s^{th} stage of Bockstein spectral sequence, we get

$$0 = e^*(\beta_s(v_3)) = \beta_s e^*(v_3) = \beta_s(b_3)$$

and then the equation

$$\begin{aligned} a_2a_4 = e^*(v_6) = e^*(\beta_s(v_5)) = \beta_s e^*(v_5) = \beta_s(b_1a_4 + b_3a_2) = \\ = \beta_s(b_1)a_4 + b_1\beta_s(a_4) + \beta_s(b_3)a_2 + b_3\beta_s(a_2). \end{aligned}$$

implies $\beta_s(b_1) = a_2$. At the $(s+1)^{th}$ stage, we get

$$a_2^2 + a_4 = e^*(v_4) = e^*(\beta_{s+1}(v_3)) = \beta_{s+1}(e^*(v_3)) = \beta_{s+1}(b_3).$$

Therefore $\beta_{s+1}(b_3) = a_4$. Also in this case it follows that $H^*_\beta(\mathrm{Map}(BV, X)_{\bar f_V}) \cong H^*_\beta(BGL_2(\mathbb{F}_q))$. By Section 3, the space $\mathrm{Map}(BV, X)_{\bar f_V}$ is homotopy equivalent to $BGL_2(\mathbb{F}_q)^\wedge_2$. □

Let us define $f'_V \colon \mathrm{Map}(BV, X)_{\bar f_V} \longrightarrow X$ to be the evaluation map, where $\bar f_V$ is the map defined in the previous proposition, and let f_V be the composite of 2-completion $BGL_2(\mathbb{F}_q) \longrightarrow BGL_2(\mathbb{F}_q)^\wedge_2$ and the map f'_V. By the above proposition f_V^* equals the composite $H^*_\beta(X) \cong H^*_\beta(BSL_3(\mathbb{F}_q)) \xrightarrow{Bi^*_{GL_2(\mathbb{F}_q)}} H^*_\beta(BGL_2(\mathbb{F}_q))$. We obtain the following diagram

$$\begin{array}{ccc} BGL_2(\mathbb{F}_q) & \overset{\Sigma_3/\Sigma_2}{\Longleftarrow} & (B(\mathbb{Z}/q-1)^2) \circlearrowleft \Sigma_3 \\ & {\scriptstyle f_V}\searrow \quad \swarrow{\scriptstyle f_W} & \\ & X & \end{array}$$

The diagram commutes on the level of mod-2 cohomology and therefore, by Lannes' theory, it commutes up to homotopy. Hence the diagram is a natural transformation $f \colon \alpha \longrightarrow \mathcal{X}$, defined only up to homotopy, from the category α to the constant category $\mathcal{X}$. The diagram induces a map from the 1-skeleton of $\mathrm{hocolim}_{\mathbb{A}}\, \alpha$ to X. Obstructions for extending this map to the whole $\mathrm{hocolim}_{\mathbb{A}}\, \alpha$ lie in $\lim^{j+1}_{\mathbb{A}} \pi_j(\mathrm{Map}(\alpha, X)_f)$ for $j \geq 1$ [22]. By lemma 4.2 below, the obstruction groups vanish, hence there exists a map $f\colon \mathrm{hocolim}_{\mathbb{A}}\, \alpha \longrightarrow X$. By construction of the map f, the diagram

$$\begin{array}{ccc} BGL_2(\mathbb{F}_q)^\wedge_2 & & \\ \downarrow & \searrow^{f_V} & \\ BSL_3(\mathbb{F}_q)^\wedge_2 & \xrightarrow{f} & X \end{array}$$

commutes up to homotopy. Because f_V^* is a monomorphism, the same is true for the map f^*, and therefore f^* is an isomorphism. This shows that $f_2^\wedge \colon BSL_3(\mathbb{F}_q)^\wedge_2 \longrightarrow X$ is a homotopy equivalence.

Lemma 4.2. *For $j \geq 1$, define a functor $\Pi_j \colon \mathbb{A}^{op} \longrightarrow Ab$ as*

$$\Pi_j(U) = \pi_j(\mathrm{Map}(BC_{SL_3(\mathbb{F}_q)}(U), X)_{f_U}).$$

Then $\lim^{j+1}_{\mathbb{A}} \Pi_j = 0$ for all $j \geq 1$.

Proof. By [6, Proposition 10.3], there is a long exact sequence

$$0 \longrightarrow \lim^0_{\mathbb{A}} \Pi_j \longrightarrow \Pi_j(V) \longrightarrow \Pi_j(W)^{\Sigma_2}/\Pi_j(W)^{\Sigma_3} \longrightarrow \lim^1_{\mathbb{A}} \Pi_j \longrightarrow$$
$$\longrightarrow H^1(\Sigma_3; \Pi_j(W)) \longrightarrow H^1(\Sigma_2; \Pi_j(W)) \longrightarrow \lim^2_{\mathbb{A}} \Pi_j \longrightarrow H^2(\Sigma_3; \Pi_j(W)) \longrightarrow \cdots$$

By the Shapiro lemma [5, Ch. III, Proposition 6.2], $H^*(\Sigma_2; (\mathbb{Z}/2)^2) = H^*(1; \mathbb{Z}/2)$. By a transfer argument, $H^*(\Sigma_3; (\mathbb{Z}/2)^2)$ is a subgroup in $H^*(\Sigma_2; (\mathbb{Z}/2)^2)$. It follows that

$$H^n(\Sigma_3; (\mathbb{Z}/2)^2) = H^n(\Sigma_2; (\mathbb{Z}/2)^2) = H^n(1; \mathbb{Z}/2) = 0$$

for $n \geq 1$. If we insert this in the above long exact sequence we get $\lim^n_{\mathbb{A}} \Pi_j = 0$ for $n \geq 2$. □

5. Outher automorphism group $\mathrm{Out}(BSL_3(\mathbb{F}_q)^\wedge_2)$

In the next section we will prove homotopy uniqueness of $BGL_3(\mathbb{F}_q)^\wedge_2$ with the strategy that we used for the proof of mod-2 determinism of $BGL_2(\mathbb{F}_q)$ in Section 3. We will investigate all possible fibrations of the form $BSL_3(\mathbb{F}_q)^\wedge_2 \longrightarrow X \longrightarrow B\mathbb{Z}/2^s$, where $2^s \| q-1$, and prove that only one X in such a fibration has the same mod-2 cohomology as $BGL_3(\mathbb{F}_q)$. In order to do that we need to determine all possible actions $\mathbb{Z}/2^s \longrightarrow \mathrm{Out}(BSL_3(\mathbb{F}_q)^\wedge_2)$. In this section we will calculate the group $\mathrm{Out}(BSL_3(\mathbb{F}_q)^\wedge_2)$.

Let G be a finite group. A p-subgroup P of G is p-centric if its center $Z(P)$ is a p-Sylow subgroup of the centralizer $C_G(P)$. Furthermore P is p-radical if the quotient group $N_G(P)/P$ is p-reduced, which means that it does not have nontrivial normal p-subgroups. Let S be a p-Sylow subgroup of G. Then S is a p-centric p-radical subgroup. Let $S = P_0, P_1, \ldots, P_m$ denote a choice of G-conjugacy class representatives for all p-centric p-radical subgroups of G contained in S. We write $N'(P_i) = N_G(P_i)/C'_G(P_i)$, where $C'_G(P_i)$ is the p'-torsion in the centralizer $C_G(P_i)$. Let $X(G)$ be the set of all $(m+1)$-tuples $(\theta; \theta_1, \ldots, \theta_m)$ such that

$$\theta \colon N'(S) \xrightarrow{\cong} N'(S) \qquad \text{and} \qquad \theta_i \colon N'(P_i) \xrightarrow{\cong} N'(\theta(P_i))$$

are isomorphisms, and such that θ_i and θ restricted to the image of $N_G(S) \cap N_G(P_i)$ in $N'(P_i)$ are equal for all i. The group $N'(S)$ acts on $X(G)$ by

$$x \cdot (\theta; \theta_1, \ldots, \theta_m) = (c_x \circ \theta; c_x \circ \theta_1, \ldots, c_x \circ \theta_m),$$

where c_x is conjugation by the element x. If there are no i, j with $1 \leq i, j \leq m$ such that P_i is conjugate to a proper subgroup of P_j, then $X(G)/N'(S)$ is isomorphic to $\mathrm{Out}(BG^\wedge_p)$ [9, Proposition 6.3 and Theorem B].

Theorem 5.1. *If $q \equiv 1 \pmod 4$ let s be such that $2^s \| q-1$, and if $q \equiv 3 \pmod 4$ let s be such that $2^s \| q+1$. Then $\mathrm{Out}(BSL_3(\mathbb{F}_q)^\wedge_2)$ is isomorphic to $\mathbb{Z}/2^{s-1}$.*

Proof. Let $q \equiv 1 \pmod 4$. Let ξ' be a generator of $\mathbb{F}_q^*$. Then $\xi = (\xi')^{\frac{q-1}{2^s}}$ is a generator of $\mathbb{Z}/2^s < \mathbb{F}_q^*$. Define the following matrices in $SL_3(\mathbb{F}_q)$:

$$Z' = \begin{bmatrix} \xi' & 0 & 0 \\ 0 & \xi' & 0 \\ 0 & 0 & (\xi')^{-2} \end{bmatrix}, A = \begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi^{-1} \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and $Z = (Z')^{\frac{q-1}{2^s}}$. Then $S = \langle A, B\rangle$ is a 2-Sylow subgroup of $SL_3(\mathbb{F}_q)$. Let $P < S$ be a 2-centric 2-radical subgroup of $SL_3(\mathbb{F}_q)$. Because P is 2-centric, the center $\langle Z\rangle$ of S is a subgroup of P. If P is a subgroup of the group $T = \langle Z, A\rangle$ of the diagonal matrices of S, then $P = T$ is the only candidate to be a 2-centric 2-radical subgroup of $SL_3(\mathbb{F}_q)$. The normalizer $N(T)$ equals $\langle A, Z', D\rangle$, where D is the permutation matrix that corresponds to the permutation $(1, 2, 3)$. Hence $N(T)/T$ is 2-reduced, and therefore T is 2-radical.

Every element in $S - T$ is conjugate to Z^iB or Z^iAB for some i. Hence, if P is not a subgroup of T (and contains $\langle Z\rangle$), then P is conjugate to one of the groups $P_i = \langle Z, A^{2^i}, B\rangle$ or $Q_i = \langle Z, A^{2^i}, AB\rangle$ for $0 \le i \le s$. The groups P_0 and Q_0 equal the 2-Sylow group S. The group P_s is subconjugate to the group T, so it is not 2-centric. It is easy to see that

$$\begin{aligned} N(P_i) &= \langle Z', A^{2^{i-1}}, B\rangle \text{ for } 0 < i < s-1, \\ N(Q_i) &= \langle Z', A^{2^{i-1}}, AB\rangle \text{ for } 0 < i < s, \\ N(P_{s-1}) &= \langle Z', A^{2^{s-2}}, B, C\rangle, \\ N(Q_s) &= \langle D, A^{2^{s-1}}\rangle, \end{aligned}$$

where

$$C = \begin{bmatrix} (-1+\xi^{2^{s-2}})2^{-1} & (-1-\xi^{2^{s-2}})2^{-1} & 0 \\ (1-\xi^{2^{s-2}})2^{-1} & (-1-\xi^{2^{s-2}})2^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and D is a generator of the centralizer $C_{SL_3(F_q)}(Q_s) = \mathbb{Z}/(q^2-1)$. So if P equals P_i or Q_i for $i > 1$, then $N'(P)/P \cong \mathbb{Z}/2$ except for the groups P_{s-1} and Q_s. Because $N(P_{s-1})/P_{s-1} \cong \mathbb{Z}/(\frac{q-1}{2^s}) \times \Sigma_3$ and $N(Q_s)/Q_s \cong \langle x, y \mid x^{\frac{q^2-1}{2^{s+1}}} = y^2, xyx = x^q\rangle$, there are exactly four 2-centric 2-radical subgroups of $SL_3(\mathbb{F}_q)$, namely S, T, P_{s-1}, and Q_s.

Because $N(S) = \langle A, B, Z'\rangle$, $N'(S) \cong S = \langle a, b, c \mid a^{2^s} = b^{2^s} = c^2 = 1, ab = ba, cac = b\rangle$. Every automorphism $N'(S) \longrightarrow N'(S)$ is of the form $a \mapsto a^\alpha b^\beta$ and $c \mapsto a^\gamma b^{-\gamma}c$ for $\alpha+\beta$ an odd number. Denote this automorphism by $\theta(\alpha, \beta, \gamma)$.

The normalizer $N(T)$ equals $\langle A, Z', D\rangle$, hence

$$N'(T) = \langle a, b, c, d \mid a^{2^s} = b^{2^s} = c^2 = d^3 = 1, ab = ba, cac = b, \\ dad^{-1} = b^{-1}, d^{-1}ad = ba^{-1}, cdc = d^2\rangle.$$

We can extend $\theta(\alpha, \beta, \gamma)$ to an automorphism $N'(T) \longrightarrow N'(T)$ only if $\alpha = 0$ or $\beta = 0$. In case $\beta = 0$, an automorphism $\theta_T(\alpha, \gamma)\colon N'(T) \longrightarrow N'(T)$ maps d to

$a^\gamma b^{-\gamma} d$ and in case $\alpha = 0$ an automorphism $\theta_T(\beta,\gamma)\colon N'(T) \longrightarrow N'(T)$ maps d to $a^\gamma b^{-\gamma} d^2$.

Since $c_{a^x b^y} \circ \theta(1,0,0) = \theta(1,0,x-y)$ and $c_{a^x b^y c} \circ \theta(1,0,0) = \theta(0,1,x-y)$, only elements of the center of $SL_3(\mathbb{F}_q)$ fix the automorphism $\theta(1,0,0)$. Every $\theta(\alpha,\beta,\gamma)$ which has an extension to an automorphism of $N'(T)$ has the form $\theta(\alpha,0,0)\circ c_g\theta(1,0,0)$ for some α and g. Hence we may take care only of the automorphisms $\theta(\alpha,0,0)$ where α is an odd number.

Because

$$\begin{aligned} N'(P_{s-1}) = \langle \bar{a}, \bar{b}, c, e \mid \bar{a}^4 = \bar{b}^4 = c^2 = e^3 = 1,& \\ \bar{a}\bar{b} = \bar{b}\bar{a}, c\bar{a}c = \bar{b}, \bar{a}e\bar{a}^{-1} = \bar{a}^3\bar{b}e^2, cdc = \bar{a}\bar{b}^3 e\rangle,& \end{aligned}$$

the intersection $N'(S) \cap N'(P_{s-1})$ is generated by $\bar{a}$, $\bar{b}$, and c. There is only one extension $\theta_P(\alpha)$ of the morphism $\theta(\alpha,0,1)|_{N'(S)\cap N'(P_{s-1})}$ to a morphism of $N'(P_{s-1})$. In case $\alpha \equiv 1 \pmod 4$, the morphism $\theta(\alpha,0,0)$ maps $\bar{a}$ to $\bar{a}$, so $\theta_P(\alpha)$ maps e to e. If $\alpha \equiv 3 \pmod 4$, the morphism $\theta(\alpha,0,0)$ maps $\bar{a}$ to $\bar{a}^3$, hence $\theta_P(\alpha)$ maps e to $\bar{a}^3\bar{b}^3ce$.

Because $N'(Q_s) < N'(S)$, we can omit the group Q_s, hence

$$X(SL_3(\mathbb{F}_q))/N'(S) = \{\Theta(\alpha) := (\theta(\alpha,0,0); \theta_T(\alpha,0), \theta_P(\alpha)) \mid \alpha \text{ is odd}\}$$

and $\Theta(\alpha)\Theta(\beta) = \Theta(\alpha\cdot\beta)$, so $\mathrm{Out}(BSL_3(\mathbb{F}_q)^{\wedge}_2) \cong \mathbb{Z}/2^{s-1}$.

Let $q \equiv 3 \pmod 4$. Let ξ' be a generator of $\mathbb{F}^*_{q^2}$. Then $\xi = (\xi')^{\frac{q^2-1}{2^{s+1}}}$ is a generator of $\mathbb{Z}/2^{s+1} < \mathbb{F}^*_{q^2}$, and $\zeta = (\xi')^{q+1}$ is a generator of $\mathbb{F}^*_q$. Let

$$P = \begin{bmatrix} 1 & 1 & 0 \\ \xi & -\xi^{-1} & 0 \\ 0 & 0 & (-\xi-\xi^{-1})^{-1} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 & \xi^{-2} & 0 \\ \xi^2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$\bar{A} = \mathrm{Diag}(\xi, -\xi^{-1}, -1)$, and $\bar{Z} = \mathrm{Diag}(\xi', (\xi')^q, (\xi')^{-q-1})$. Then the matrices $A = P\bar{A}P^{-1}$ and $B = P\bar{B}P^{-1}$ generate a 2-Sylow subgroup S of $SL_3(\mathbb{F}_q)$ [10, Lemma 1] and the $(q+1)^{th}$ power of $Z = P\bar{Z}P^{-1}$ is a generator of the center $Z(SL_3(\mathbb{F}_q))$. Let $P < S$ be a 2-centric 2-radical subgroup of $SL_3(\mathbb{F}_q)$. If P is a subgroup of $T = \langle A\rangle$, then $P = T$ is the only candidate for a 2-centric 2-radical subgroup of $SL_3(\mathbb{F}_q)$. The normalizer $N(T)$ equals $\langle B, Z\rangle$, hence $N(T)/T = \langle x, y \mid x^{\frac{q^2-1}{2^{s+1}}} = y^2 = 1, yxy = x^q\rangle$ is 2-reduced if and only if $\langle [B]\rangle$ is normal subgroup of $N(T)/T$, and this is true if and only if $q+1 = 2^s$.

Every element in $S - T$ is conjugate to B, AB, $I'B$ or $I'AB$, where $I' = A^{2^s} = \mathrm{Diag}(-1,-1,1)$. Because I' is in the center of S, it follows that I' is in every 2-centric subgroup P of S. Hence, if P is not a subgroup of T, then P is conjugate to one of the groups $P_i = \langle A^{2^i}, B\rangle$ or $Q_i = \langle A^{2^i}, AB\rangle$ for $0 \le i \le s$. Then

$$\begin{aligned} N(P_i) &= \langle Z^{q+1}, A^{2^{i-1}}, B\rangle \text{ for } 0 < i < s, \\ N(Q_i) &= \langle Z^{q+1}, A^{2^{i-1}}, AB\rangle \text{ for } 0 < s-1, \\ N(P_s) &= \langle A^{2^{s-1}}, C_1, C_2, D\rangle, \\ N(Q_{s-1}) &= \langle A^{2^{s-2}}, AB, E\rangle, \end{aligned}$$

where

$$C_1 = \begin{bmatrix} \zeta & (\zeta-1)(\xi-\xi^{-1})2^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{-1} \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & (1-\zeta)(\xi-\xi^{-1})2^{-1} & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 & (\xi^{-1}-\xi)2^{-1} \\ 0 & 0 & 1 \\ 1 & (\xi-\xi^{-1})2^{-1} & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1+\xi^{2^{2-1}} & (1+\xi^{2^{2-1}})\xi^{-1} & 0 \\ (-1+\xi^{2^{2-1}})\xi & 1-\xi^{2^{2-1}} & 0 \\ 0 & 0 & 4^{-1} \end{bmatrix}.$$

The groups P_0 and Q_0 equal S, the group $Q_s \cong \mathbb{Z}/4$ is not 2-centric and the group Q_{s-1} is isomorphic to the quaternion group $Q(8)$. Hence 2-centric 2-radical subgroups of $SL_3(\mathbb{F}_q)$ are S, P_s, and Q_{s-1}, and if $q+1$ is a power of 2 then also T.

Because $N'(S) \cong S = \langle a, b \mid a^{2^{s+1}} = b^2 = 1, bab = -a^{-1}\rangle$, every automorphism $N'(S) \longrightarrow N'(S)$ is of the form $a \mapsto a^\alpha$ and $b \mapsto a^\beta b$ for α an odd number and β an even number. Denote this automorphism by $\theta(\alpha, \beta)$.

Since $c_{a^{2x}} \circ \theta(1,0) = \theta(1, 4x)$, $c_{a^{2x+1}} \circ \theta(1,0) = \theta(1, 4x+2+2^s)$, $c_{a^{2x}b} \circ \theta(1,0) = \theta(2^s-1, 4x)$, and $c_{a^{2x+1}b} \circ \theta(1,0) = \theta(2^s-1, 4x+2+2^s)$, only elements of the center of S fix the automorphism $\theta(1,0)$. Because every $\theta(\alpha,\beta)$ equals $\theta(\alpha',0)\cdot c_x\theta(1,0)$, for some x and some $\alpha' \equiv 1 \pmod 4$, we may take care only of the automorphisms $\theta(\alpha, 0)$ where $\alpha \equiv 1 \pmod 4$.

Because $C_1^{\frac{q-1}{2}} = I'B$ and $C_2^{\frac{q-1}{2}} = B$, it follows that $N'(S) \cap N'(P_s) = \langle B, I', A^{2^{s-1}}\rangle$. The morphism $\theta(\alpha,0)|_{N'(S)\cap N'(P_s)}$ has two extensions to an automorhism of $N'(P_s)$. Let d and i' be the images of D and I' in $N'(P_s)$. If $\alpha \equiv 1 \pmod 4$ then the first extension $\theta_P^1(\alpha)$ maps d to d and the second one $\theta_P^2(\alpha)$ maps d to $i'bd$. If $\alpha \equiv 3 \pmod 4$ then $\theta_P^1(\alpha)$ maps d to $i'd$ and $\theta_P^2(\alpha)$ maps d to bd. The extensions are connected by conjugation by the element $i' = a^{2^{s-1}}$, i.e. $c_{i'} \circ \theta_P^1 = \theta_P^2$. Note that the conjugation $c_{i'}$ fixes any morphism $\theta(\alpha, \beta)$.

The morphism $\theta(\alpha,0)|_{N'(S)\cap N'(Q_{s-1})}$ has only one extension to an automorphism of $N'(Q_{s-1})$. Let e be the image of E in $N'(Q_{s-1})$. If $\alpha \equiv 1 \pmod 4$ then the extension $\theta_Q(\alpha)$ maps e to e and if $\alpha \equiv 3 \pmod 4$ then $\theta_Q(\alpha)$ maps e to $a^{-2^{s-1}}d$. The morphism $\theta_Q(\alpha)$ is fixed by conjugation by the element i'.

Because $N(T) = N(S)$, we can omit this group even if $q+1$ is a power of 2. So

$$X(SL_3(\mathbb{F}_q))/N'(S) = \{\Theta(\alpha) := (\theta(\alpha,0); \theta_P^1(\alpha,0), \theta_Q(\alpha)) \mid \alpha \equiv 1 \pmod 4\},$$

$\Theta(\alpha)\Theta(\beta) = \Theta(\alpha\cdot\beta)$, and therefore $\mathrm{Out}(BSL_3(\mathbb{F}_q)_2^\wedge) \cong \mathbb{Z}/2^{s-1}$. □

The following technical lemma will be used in the next section.

Lemma 5.2. *Let $\psi \in \mathrm{Out}(BSL_3(\mathbb{F}_{p^n})_2^\wedge)$ be a nontrivial automorphism and let $2^t \| p^{2n}-1$. Then 2^t does not divide the order of the fixed-point set $H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\frac{1}{p}])^\psi$.*

Proof. We define inclusion $i\colon \mathbb{Z}/2^{t-1} \longrightarrow SL_3(\mathbb{F}_{p^n})$ depending upon p as follows: $i(\zeta^k) = (\mathrm{Diag}(\xi, 1, \xi^{-1}))^k$, if $p^n \equiv 1 \pmod 4$ and ξ is a generator of $\mathbb{Z}/2^{t-1} < \mathbb{F}_{p^n}^*$, and $i(\zeta^k) = P(\mathrm{Diag}(\xi, \xi^{p^n}, \xi^{-1-p^n}))^k P^{-1}$, if $p^n \equiv 3 \pmod 4$, ξ is a generator of $\mathbb{Z}/2^t < \mathbb{F}_{p^{2n}}^*$ and P the matrix defined in the proof of the previous theorem. Let $x \in \mathbb{Z}/2^t$ considered as a subgroup of $H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\frac{1}{p}])$ [17, Theorem 2.3] be a generator. Then $i^*(x)$ is a generator of $H^4(B\mathbb{Z}/2^{t-1}; \mathbb{Z}[\frac{1}{p}]) = \mathbb{Z}/2^{t-1}$. By the proof of the previous theorem $\psi = \Theta(\alpha)$ and because ψ is a nontrivial automorphism, it follows that $\alpha \neq 1$. So the restriction of $\Theta(\alpha)$ to the subgroup $\mathbb{Z}/2^{t-1}$ is nontrivial, hence $i^*(x)$ is not fixed by the restriction map, so also x is not fixed by $\Theta(\alpha)$, which means that 2^{t-1} does not divide the order of $H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\frac{1}{p}])^\psi$. □

6. The Homotopy type of $BGL_3(\mathbb{F}_q)_2^\wedge$

The group $GL_3(\mathbb{F}_q)$ has order $q^3(q-1)^3(q^2+q+1)(q+1)$. If $q \equiv 3 \pmod 4$, the cohomology of $BGL_3(\mathbb{F}_q)$ is

$$H^*(BGL_3(\mathbb{F}_q)) = \mathbb{F}_2[b_1, b_3, a_4, b_5]/(b_1^4 b_3^2 + b_1^6 a_4 + b_3^2 a_4 + b_5^2), \tag{5}$$

and the action of the Steenrod algebra is defined as follows:

	b_1	b_3	a_4	b_5
Sq^1	b_1^2	b_1^4	0	b_3^2
Sq^2	0	b_5	$b_1^6 + b_3^2$	0
Sq^4	0	0	a_4^2	$b_1^9 + b_1^6 b_3 + b_1^5 a_4 + b_1^4 b_5 + b_1^3 b_3^2 + b_3^3 + a_4 b_5$

and $\beta_{s+1}(b_1^3 + b_3) = a_4$, where $2^s || (q+1)$ ([13, IV, Theorem 8.2] and [17, Theorem 1.3, Theorem 2.3]). If we change the generators b_3 and b_5 by respectively $b_3 + b_1^3$ and $b_5 + b_1^5$, we see that $H^*_\beta(BGL_3(\mathbb{F}_q))$ and $H^*_\beta(BSL_3(\mathbb{F}_q)) \otimes H^*_\beta(B\mathbb{Z}/q-1)$ are isomorphic as objects in the category $\mathcal{K}_\beta$.

If $q \equiv 1 \pmod 4$ then

$$H^*(BGL_3(\mathbb{F}_q)) = \mathbb{F}_2[a_2, a_4, a_6] \otimes E(b_1, b_3, b_5), \tag{6}$$

and the action of the Steenrod algebra is defined as follows:

	b_1	a_2	b_3	a_4	b_5	a_6
Sq^1	0	0	0	0	0	0
Sq^2	0	a_2^2	$b_1 a_4 + b_3 a_2 + b_5$	$a_2 a_4 + a_6$	$b_1 a_6 + b_5 a_2$	$a_2 a_6$
Sq^4	0	0	0	a_4^2	$b_3 a_6 + b_5 a_4$	$a_4 a_6$

and $\beta_s(b_1) = a_2$, $\beta_{s+1}(b_3) = a_4$ and $\beta_s(b_5) = a_6$, where $2^s || (q-1)$ ([13, IV Theorem 8.1] and [17, Theorem 1.3, Theorem 2.3]). If we change the generators a_4, a_6, and b_5 by respectively $a_4 + a_2^2$, $a_2 a_4 + a_6$, and $b_1 a_4 + b_3 a_2 + b_5$, we see that $H^*_\beta(BGL_3(\mathbb{F}_q))$ is isomorphic to $H^*_\beta(BSL_3(\mathbb{F}_q)) \otimes H^*_\beta(B\mathbb{Z}/q-1)$ as an object in $\mathcal{K}_\beta$.

Let X be a 2-complete space and $H^*_\beta(X) \cong H^*_\beta(BGL_3(\mathbb{F}_q))$. Let $2^s || (q-1)$ and let $g\colon X \longrightarrow B\mathbb{Z}/2^s$ be a map such that g^* maps the generator of $H^1(B\mathbb{Z}/2^s)$ to the generator of $H^1(X)$. Let Y be the homotopy fiber of the map g. Using the Eilenberg-Mooer spectral sequence, we see that $H^*_\beta(Y) \cong H^*_\beta(BSL_3(\mathbb{F}_q))$. Hence Y is homotopy equivalent to $BSL_3(\mathbb{F}_q)^\wedge_2$ (Section 4).

Let $\alpha\colon B\mathbb{Z}/2^s \longrightarrow B\,\mathrm{Out}(BSL_3(\mathbb{F}_q)^\wedge_2)$ be the action induced by the fibration $BSL_3(\mathbb{F}_q)^\wedge_2 \longrightarrow Y \longrightarrow B\mathbb{Z}/2^s$. Let $O_{2'}(SL_3(\mathbb{F}_q))$ be the maximal normal subgroup of $SL_3(\mathbb{F}_q)$ of order prime to 2. Then $O_{2'}(SL_3(\mathbb{F}_q))$ is the subgroup of diagonal matrices. Fibrations of the form $BSL_3(\mathbb{F}_q)^\wedge_2 \longrightarrow Y \longrightarrow B\mathbb{Z}/2^s$ with the specified action are in bijection with $H^2(B\mathbb{Z}/2^s; Z(SL_3(\mathbb{F}_q)/O_{2'}(SL_3(\mathbb{F}_q))))$ (see [8]). Because the center $Z(SL_3(\mathbb{F}_q)/O_{2'}(SL_3(\mathbb{F}_q)))$ is trivial, there exists exactly one such fibration. We will show that the total space Y has the mod-2 cohomology isomorphic to that of $BGL_3(\mathbb{F}_q)$ only if Y induces the trivial action $B\mathbb{Z}/2^s \longrightarrow B\,\mathrm{Out}(BSL_3(\mathbb{F}_q)^\wedge_2)$. To do this we employ similar methods as in the Section 3.

Let $q = p^n$. Because $H^j(BSL_3(F_{p^n}); \mathbb{Z}[\frac{1}{p}]) = 0$ for $j = 1, 2, 3$ [17, Theorem 2.3], the elements $E_2^{i,j}$ of the Serre spectral sequence of the fibration $BSL_3(\mathbb{F}_{p^n})^\wedge_2 \longrightarrow Y \longrightarrow B\mathbb{Z}/2^s$ vanish for $j = 1, 2, 3$. And also $E_2^{5,0} = H^5(B\mathbb{Z}/2^s; H^0(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\frac{1}{p}])) = 0$, hence

$$\begin{aligned} H^4(Y; \mathbb{Z}[\tfrac{1}{p}]) &= E_2^{4,0} \oplus E_2^{0,4} = \\ &= H^4(\mathbb{Z}/2^s; H^0(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\tfrac{1}{p}])) \oplus H^0(\mathbb{Z}/2^s; H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\tfrac{1}{p}])) = \\ &= \mathbb{Z}/2^s \oplus H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\tfrac{1}{p}])^{\mathbb{Z}/2^s}, \end{aligned}$$

where $H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\frac{1}{p}])^{\mathbb{Z}/2^s}$ is the fixed-point set of the action α. By lemma 5.2, if $2^t \| p^{2n} - 1$ then 2^{t+1} does not divide the order of $H^4(BSL_3(\mathbb{F}_{p^n}); \mathbb{Z}[\frac{1}{p}])^{\mathbb{Z}/2^s}$, so there are no elements in $H^4(Y; \mathbb{F}_2)$ which are maped nontrivally by β_t. This implies that the mod-2 cohomology of Y differs from the mod-2 cohomology of $BGL_3(\mathbb{F}_{p^n})$ if the fibration $BSL_3(\mathbb{F}_q)^\wedge_2 \longrightarrow Y \longrightarrow B\mathbb{Z}/2^s$ induces a nontrivial action.

Corollary 6.1. *Let q be a power of an odd prime. The space $BGL_3(\mathbb{F}_q)^\wedge_2$ is homotopy equivalent to the product $BSL_3(\mathbb{F}_q)^\wedge_2 \times (B\mathbb{Z}/q-1)^\wedge_2$.*

7. The Mathieu group M_{11}

The Mathieu group M_{11} has the same cohomology as the group $SL_3(\mathbb{F}_3)$ as an object in $\mathcal{K}_\beta$ [2, Section 12]. Hence by theorem 1.1, we recover the following result, which is due to J. Martino and S. Priddy [20, Theorem 4].

Corollary 7.1. *The 2-completions of the classifying spaces $(BM_{11})^\wedge_2$ and $BSL_3(\mathbb{F}_3)^\wedge_2$ are homotopy equivalent.*

This result allows us to prove the following theorem.

Theorem 7.2. *There exists a map $f\colon BM_{11} \longrightarrow BSU(3)$ inducing an injective map $f^*\colon H^*(BSU(3)) \longrightarrow H^*(BM_{11})$. The mod-2 cohomology of BM_{11} is a finitely generated free module over the image of f^*.*

Let us look at the tower

$$BSL_3(\mathbb{F}_3) \xrightarrow{Bi_1} BSL_3(\mathbb{F}_{3^2}) \xrightarrow{Bi_2} BSL_3(\mathbb{F}_{3^{2^2}}) \longrightarrow \cdots \xrightarrow{Bi_n} BSL_3(\mathbb{F}_{3^{2^n}}) \longrightarrow \cdots$$

where the maps Bi_n are induced by inclusions $i_n\colon \mathbb{F}_{3^{2^{n-1}}} \longrightarrow \mathbb{F}_{3^{2^n}}$. For $n \geq 2$ the cohomology is $H^*(BSL_3(\mathbb{F}_{3^{2^{n-1}}})) = \mathbb{F}_2[y_4^{(n)}, y_6^{(n)}] \otimes E(x_3^{(n)}, x_5^{(n)})$ and the map Bi_n^* is defined by $Bi_n^*(y_4^{(n)}) = y_4^{(n-1)}$, $Bi_n^*(y_6^{(n)}) = y_6^{(n-1)}$, $Bi_n^*(x_3^{(n)}) = 0$, and $Bi_n^*(x_5^{(n)}) = 0$. Then the cohomology of the colimit of the tower is

$$H^*(\varinjlim BSL_3(\mathbb{F}_{3^{2^n}})) = \varprojlim H^*(BSL_3(\mathbb{F}_{3^{2^n}})) = \mathbb{F}_2[y_4, y_6],$$

and this is isomorphic to the cohomology $H^*(BSU(3))$. Because $BSU(3)_2^\wedge$ is determined by cohomology [21], $(\varinjlim BSL_3(\mathbb{F}_{3^{2^n}}))_2^\wedge \simeq BSU(3)_2^\wedge$. Hence there exists a map $(BM_{11})_2^\wedge \simeq BSL_3(\mathbb{F}_3)_2^\wedge \longrightarrow BSU(3)_2^\wedge$ and by the theorem of W. Dwyer and C. Wilkerson [11, Proposition 3.1], there exists a map $f\colon BM_{11} \longrightarrow BSU(3)$.

The cohomology of the first space in the tower is

$$H^*(BSL_3(\mathbb{F}_3)) = \mathbb{F}_2[v_3, v_4, v_5]/(v_3^2 v_4 + v_5^2)$$

and the map Bi_1^* is defined as $Bi_1^*(y_4^{(2)}) = v_4$, $Bi_1^*(y_6^{(2)}) = v_3^2$, $Bi_1^*(x_3^{(2)}) = 0$, and $Bi_1^*(x_5^{(2)}) = 0$, therefore

$$f^*\colon H^*(BSU(3)) = \mathbb{F}_2[y_4, y_6] \longrightarrow H^*(BM_{11}) = \mathbb{F}_2[v_3, v_4, v_5]/(v_3^2 v_4 + v_5^2)$$

is given by $f^*(y_4) = v_4$ and $f^*(y_6) = v_3^2$, hence $H^*(BM_{11})$ is a finitely generated $H^*(BSU(3))$ module.

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